

Non-invertible symmetry-protected topological order in a group-based cluster state

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(Dated: December 18, 2023)

Despite growing interest in beyond-group symmetries in quantum condensed matter systems, there are relatively few microscopic lattice models explicitly realizing these symmetries, and many phenomena have yet to be studied at the microscopic level. We introduce a one-dimensional stabilizer Hamiltonian composed of group-based Pauli operators whose ground state is a $G \times \text{Rep}(G)$ -symmetric state: the G cluster state introduced in [Brell, *New Journal of Physics* **17**, 023029 (2015)]. We show that this state lies in a symmetry-protected topological (SPT) phase protected by $G \times \text{Rep}(G)$ symmetry, distinct from the symmetric product state by a duality argument. We identify several signatures of SPT order, namely protected edge modes, string order parameters, and topological response. We discuss how G cluster states may be used as a universal resource for measurement-based quantum computation, explicitly working out the case where G is a semidirect product of abelian groups.

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I. INTRODUCTION

Symmetry is a central organizing principle and computational tool in the study of condensed matter physics. Conventionally, we think of a family of symmetry operators which:

1. act on the full system, e.g., as a tensor product “on-site” (internal) symmetry or by realizing a transformation, such as rotation or inversion, of the underlying lattice; and
2. multiply according to the multiplication rule of some group.

However, this is not the most general situation. Exotic phenomena can be described by symmetries which violate one or both of the above criteria. In loosening the definition of symmetry, we are able to unify a wider family of quantum phases within a single framework and explore novel phases which pose challenges of theoretical and practical interest (see e.g. [1, 2]).

Rich physics has emerged from the study of two particular generalizations of global symmetry which act on

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only part of the system and therefore relax the first constraint above — higher form symmetry [3–6] and subsystem symmetry [7–11]. Both types of symmetry act on subsets of local degrees of freedom, such as qubits or qudits, that make up manifolds of nonzero codimension in the geometry of the many-body system.

Higher form symmetries act on homologically non-trivial manifolds and are deformable within their homology class via the Gauss law of a gauge theory [9, 12, 13]. A notable 1-form symmetry operator [3] is the Wilson loop operator, which acts on a subsystem that forms a closed loop and which distinguishes the confined and deconfined phases of the \mathbb{Z}_2 gauge theory in 2+1D. Other higher form symmetries are important objects in the design of topological quantum computation schemes [14, 15].

On the other hand, subsystem symmetries act on rigid manifolds that are not deformable into one another. They have become especially important in describing fracton phases of matter, which have garnered significant interest as a particularly exotic form of topological order where quasiparticle excitations exhibit restricted mobility [9, 12, 16–22].

Higher form and subsystem symmetries demonstrate the value of understanding how symmetry can be generalized by violating the *first* criterion above. Other works have studied symmetries which violate both constraints (see e.g. [23–29]), though these symmetries are less well-understood. In this work, we study a symmetry which is complementary to the above, as it violates only the *second* criterion, balancing physical richness and tractability.

Conventionally, a symmetry in a quantum system is realized by a set of unitary operators U_g which commute with the Hamiltonian and transform as a linear representation of the symmetry group G , meaning that $U_g U_h = U_{gh}$. More generally, a set of symmetry operators $\{O_a\}$ can multiply according to the rules of a *fusion category* [30, 31] (equivalently¹, fusion ring [32, 33] or fusion algebra [34]),

$$O_a O_b = \sum_c N_{a,b}^c O_c, \quad (1)$$

where a , b , and c are simple objects in the category, and where the fusion coefficients $N_{a,b}^c \in \mathbb{Z}_{\geq 0}$ give the number of copies of each O_c appearing in the sum. We write these symmetry operators as O rather than U because they are not in general unitary.

¹ Fusion categories share fusion rules with fusion rings and algebras, but are equipped with additional structure, such as F-symbols which encode the degree of associativity of fusion of vector spaces defined within the category. The topic of constructing fusion categories from a fusion ring is called *categorification*. Because we only use the fusion rules of the category, we are ignoring these subtleties.

Where multiplication in a group yields another element in the group ($g \cdot h = gh$), multiplication in a fusion category yields a sum of objects in the category ($a \times b = \sum_c N_{a,b}^c c$). Groups are special cases of fusion categories where $N_{g,h}^k = \delta_{k,gh}$, and when we refer to fusion category symmetries we implicitly mean symmetries corresponding to fusion categories that are not groups.

The fusion category perhaps most familiar to physicists – though not necessarily by name – is $SU(2)_k$ [35–37], which describes the fusion of the first $k+1$ irreps of $SU(2)$ [38]. Physically, this describes addition of angular momentum restricted to particles of spin less than or equal to $k/2$, encoding relationships like $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. As an explicit example, consider the multiplication table of $SU(2)_2$:

\otimes	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$0 \oplus 1$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	0

$SU(2)_2$

Notice that the fusion rules of $SU(2)_2$ are beyond what can be described by a group. Most strikingly, they allow for the product of two objects to equal a sum of objects. This is a central feature of the generalization from groups to fusion categories, and leads to the existence of non-invertible simple objects, and therefore non-invertible symmetries.

The study of fusion categories – and their descendants, like unitary modular tensor categories – has been fruitful in condensed matter physics [39–49]. The foremost application is describing fusion and braiding of anyons. While abelian anyons fuse according to the rules of a group, non-abelian anyons fuse according to the rules of a fusion category [50]. Non-abelian anyons exhibit a variety of exotic physics due to their unusual fusion properties, including universality for topological quantum computation [51].

In this paper, we construct a microscopic model with fusion-category SPT order on a 1D lattice of qudits whose basis states are valued in a finite group G . Our model reduces to the well-known $\mathbb{Z}_2 \times \mathbb{Z}_2$ abelian SPT order: the one-dimensional cluster state [52, 53] when $G = \mathbb{Z}_2$. For non-abelian G , the second symmetry factor in the original abelian order generalizes to $\text{Rep}(G)$, the fusion category of representations of G . Such cases yield models protected by a combination of group and representation category symmetry which, to our knowledge, has not been considered before.

We show how microscopic signatures of SPT order extend to fusion category-symmetric systems through the study of the *group cluster state* (or G cluster state, where G is a finite group), a generalization of the familiar cluster state [54]. We contribute the following:

1. A proof of the claim in Ref. 55 that the G cluster state is an SPT protected by a fusion category symmetry.

2. An identification of microscopic signatures of SPT order in the G cluster state, including string order, edge modes, and topological response.
3. Further development of the group-based Pauli stabilizer formalism from Ref. 56, making contact with matrix-product states (MPSs) and matrix-product operators (MPOs). By studying *microscopics* of explicit models as opposed to abstractly classifying phases, we open up models with such exotic symmetries to further study and potential realization on quantum devices.
4. An algorithm for performing measurement-based quantum computation on the G cluster state for a wide class of non-abelian groups, namely those which are semidirect products of abelian groups.

A. Relation to Prior Works

This paper appears among a growing literature studying exotic phases of matter in lattice models, as well as studying categorical symmetries [57–82]. Non-invertible symmetry on the lattice has been discussed in several works [83–91], though most known lattice models of *SPTs* are protected by a conventional, subsystem [92–99], or higher-form symmetry [6, 100–102]. Ref. 85 is especially relevant, as it constructs commuting projector lattice models for TQFTs with fusion category symmetry and shows that degenerate edge modes arise in the SPT class of these models. While this work is quite general and its results broadly applicable, it does not study a model with $G \times \text{Rep}(G)$ symmetry explicitly, nor does it study concrete models in great detail, so that our work is complementary. We also note that our stabilizer model contains additional structure compared to the commuting projector models constructed therein, as a stabilizer Hamiltonian can always be expressed as a sum of commuting projectors, while the contrary is not guaranteed.

Although there are few explicit lattice constructions, there is good reason to suspect that the signatures of SPT order should extend to much more general contexts. Macroscopic descriptions of SPT orders with exotic categorical symmetries have been defined within higher category theory [58, 59]. There has also been work studying fusion category symmetries in quantum field theories which have addressed the existence of fusion category-symmetric gapped phases [60, 61] and gauging non-abelian symmetries to yield fusion categorical symmetries [103]. The question of classifying fusion category-symmetric gapped phases has also been addressed at the categorical level [60, 62]. In general, category theoretic methods are very powerful for studying generalized symmetries and the phases they admit. It has been shown that for a symmetry described by a tensor category \mathcal{C} , the exact module categories over \mathcal{C} – equivalently, the fiber functors which \mathcal{C} admits – correspond to the gapped symmetric phases with symmetry \mathcal{C} [60, 104].

In this work, we use group-based Pauli operators to define microscopic lattice models, which more closely resemble spin chains. This formalism was recently used in Ref. 56 to study gapped edges of quantum doubles. Group-based Pauli operators preserve some useful features of the stabilizer formalism [105] and are natural for programming onto quantum devices. These operators also admit a graphical description as matrix product operators which makes contact with the tensor network literature on quantum phases of matter with exotic symmetries (see eg. [106–108]).

While we intentionally use group-valued degrees of freedom to generate fusion category symmetries, they arise naturally in other areas. High energy physicists have long studied gauge fields which take values in gauge symmetry groups, and lattice gauge theories work with degrees of freedom labeled by group elements living on the lattice [109–111]. In condensed matter physics, these degrees of freedom are used to construct Kitaev’s quantum double model, a generalization of the toric code which hosts non-abelian anyons [112]. Group-valued degrees of freedom are also used in the TQFT approach to constructing microscopic lattice models of SPTs with conventional symmetry [100, 113–119].

The G cluster state which we study in this work was first introduced in Ref. 55, which generalizes cluster states to group-valued qudits, as Kitaev’s quantum double model does for the toric code. Ref. 55 conjectures that G cluster states should exhibit SPT order and be usable for measurement-based quantum computation, ideas which we establish in this paper.

We extend results from the significant body of knowledge which has emerged from the study of SPTs with conventional symmetry. We study microscopic signatures of SPT order which are well-understood in conventional 1d SPTs, including edge modes and ground state degeneracy [52, 114, 120], string order [121–125], and topological response [126–129]. We also study the G cluster state as a resource state for MBQC [130, 131], which is known to be intimately related to cluster states in particular [54, 132–134] and SPT order in general [135–137].

II. NOTATIONS AND CONVENTIONS

In this section, we introduce the notations and conventions which we use in this paper. Especially important is Sec. IIC which introduces the group-based Pauli operators we use extensively.

A. Group Theory

We denote by G a finite group which is in general non-abelian. For each $g \in G$, the inverse of g is denoted \bar{g} . The identity element is denoted e .

Irreducible representations are denoted Γ and defined according to the map $\Gamma : G \rightarrow \text{GL}_{d_\Gamma}(\mathbb{C})$, where d_Γ is

the dimension of the representation. $\Gamma(g)$ is the $d_\Gamma \times d_\Gamma$ matrix which is obtained when the function Γ is evaluated at the group element g . Being representations, the Γ satisfy

$$\Gamma(g)\Gamma(h) = \Gamma(gh), \quad (2a)$$

$$\Gamma(g)^\dagger = \Gamma(\bar{g}), \quad (2b)$$

$$\Gamma(g)^n = \Gamma(g^n), \quad (2c)$$

$$\Gamma(e) = \mathbb{1}_{d_\Gamma}. \quad (2d)$$

We denote the trivial representation $\mathbf{1}$, so that $\mathbf{1}(g) = 1$ for all g . The tensor product of irreps of a finite group is fully decomposable and therefore isomorphic to a direct sum of irreps

$$\Gamma_i \otimes \Gamma_j \simeq \bigoplus_k N_{\Gamma_i, \Gamma_j}^{\Gamma_k} \Gamma_k, \quad (3)$$

where the factor $N_{\Gamma_i, \Gamma_j}^{\Gamma_k} \in \mathbb{Z}_{\geq 0}$ is called the multiplicity and gives the number of copies of Γ_k appearing in the direct sum.

B. Matrix Product States and Operators

Matrix product states (MPS) are tensor network states in 1d that efficiently represent area law states. A general MPS is written

$$|\psi_A\rangle = \sum_{\{g_i\}} \text{Tr}[BA_{g_1}^{(1)} \cdots A_{g_N}^{(N)}] |g_1, \dots, g_N\rangle, \quad (4)$$

where $A_g^{(i)}$ is the matrix evaluated at site i when $g_i = g$ and B encodes the boundary conditions. Diagrammatically, we can express each piece of the MPS as a rank-three (i.e., three-legged) tensor

$$\sum_{g_i} A_{g_i}^{(i)} \otimes |g_i\rangle = \text{---} \bigcirc \text{---} \quad (5)$$

and the entire MPS succinctly as

$$|\psi_A\rangle = \text{---} \bigcirc \cdots \bigcirc \text{---} \quad (6)$$

Similarly, matrix product operators (MPO) are tensor network operators in 1d. A general MPO O_A is written

$$\sum_{\{g_i, g'_i\}} \text{Tr}[BA_{g_1, g'_1}^{(1)} \cdots A_{g_N, g'_N}^{(N)}] |g_1, \dots, g_N\rangle \langle g'_1, \dots, g'_N|, \quad (7)$$

where $A_{g, h}^{(i)}$ is the matrix evaluated at site i when $g_i = g$ and $g'_i = h$, and B encodes the boundary conditions.

Diagrammatically, we write

$$\sum_{g_i} A_{g_i, g'_i}^{(i)} \otimes |g_i\rangle \langle g'_i| = \text{---} \boxed{A} \text{---}$$

$$O_A = \text{---} \boxed{A} \cdots \boxed{A} \text{---}$$

Note that the trace is evaluated clockwise on the red MPO virtual level.

Sometimes, when we want to emphasize that the boundary conditions of an MPS or MPO are open, we will denote them as

$$\text{---} \bigcirc \cdots \bigcirc \text{---}, \quad \text{---} \boxed{A} \cdots \boxed{A} \text{---}$$

C. Group-Based Pauli Operators

In this section, we discuss the notation for group-based Pauli operators which will be used throughout this work, largely following the conventions of Ref. 56. Further details of this construction can be found in Refs. 55 and 56.

We work with a local Hilbert space whose basis kets are labelled by the elements of a finite group G , which we typically assume is non-Abelian. The local Hilbert space is the group algebra $\mathbb{C}[G]$, and arbitrary single-site states may be written as vectors, $|\psi\rangle = \sum_{g \in G} c_g |g\rangle$, with complex coefficients c_g .

We make extensive use of group-based Pauli operators which extend the action of the qudit Pauli operators to group-valued degrees of freedom. The group-based X -type operators are labeled by group elements g and act by group multiplication²:

$$\vec{X}_g = \sum_{h \in G} |gh\rangle \langle h| \Leftrightarrow \vec{X}_g |h\rangle = |gh\rangle, \quad (8a)$$

$$\overleftarrow{X}_g = \sum_{h \in G} |h\bar{g}\rangle \langle h| \Leftrightarrow \overleftarrow{X}_g |h\rangle = |h\bar{g}\rangle. \quad (8b)$$

Diagrammatically, we have

$$\vec{X}_g = \boxed{\vec{X}_g}, \quad \overleftarrow{X}_g = \boxed{\overleftarrow{X}_g}.$$

Observe that these operators form the left and right regular representations [138] of G : $\vec{X}_g \vec{X}_h = \vec{X}_{gh}$ and

² Elsewhere in the literature, these operators are sometimes called L_g and R_g . We also note that our arrow convention is consistent with [56] but reverses that of [55].

\mathbb{Z}_2	Qubit	Group-Valued Qudit	G
Elements $\{0, 1\}$	$ 0\rangle, 1\rangle$	$ g\rangle : g \in G$	Elements $\{g \in G\}$
Irreps $\{\Gamma_+, \Gamma_-\}$	$ \pm\rangle = \frac{1}{\sqrt{2}} (\Gamma_{\pm}(0) 0\rangle + \Gamma_{\pm}(1) 1\rangle)$	$ \Gamma_{\alpha\beta}\rangle = \sqrt{\frac{d_{\Gamma}}{ G }} \sum_g [\Gamma(g)]_{\alpha\beta} g\rangle$	Irreps $\{\Gamma \in \text{Rep}(G)\}$
Bit-flip	$\vec{X}_0 = \overleftarrow{X}_0 = \mathbb{1}$ $\vec{X}_1 = \overleftarrow{X}_1 = X$	$\vec{X}_g h\rangle = gh\rangle$ $\overleftarrow{X}_g h\rangle = h\bar{g}\rangle$	Group Multiplication
—	—	$\overleftarrow{\overleftarrow{X}}_g h\rangle = hg\bar{h}\rangle$	Conjugation
Phase ± 1	$Z_{\Gamma_+} = Z_{\Gamma_+}^\dagger = \mathbb{1}$ $Z_{\Gamma_-} = Z_{\Gamma_-}^\dagger = Z$	$\text{Tr}[Z_{\Gamma}] g\rangle = \text{Tr}[\Gamma(g)] g\rangle$ $[Z_{\Gamma}]_{\alpha\beta} g\rangle = [\Gamma(g)]_{\alpha\beta} g\rangle$	Phase $[\Gamma(g)]_{\alpha\beta}$ or $\text{Tr}[\Gamma(g)]$
Controlled X	$CX g\rangle h\rangle = g\rangle g \oplus h\rangle$	$C\vec{X} g\rangle h\rangle = g\rangle gh\rangle$ $C\overleftarrow{X} g\rangle h\rangle = g\rangle h\bar{g}\rangle$	Controlled Multiplication

TABLE I. Summary of the group-valued qudit formalism, reproduced with minor edits from Ref. 55. Note that the group and irrep bases outlined in the first two rows of the table are dual to one another and related by the non-abelian Fourier transform.

$\overleftarrow{X}_g \overleftarrow{X}_h = \overleftarrow{X}_{gh}$. In the case $G = \mathbb{Z}_d$, these operators reduce to the familiar qudit Pauli operators X , $\vec{X} = \overleftarrow{X}^\dagger = X$. The group-based X operators also satisfy the following identities:

$$\begin{aligned} (\vec{X}_g)^n &= \vec{X}_{g^n}, & (\overleftarrow{X}_g)^n &= \overleftarrow{X}_{g^n} \\ (\vec{X}_g)^\dagger &= \vec{X}_{\bar{g}}, & (\overleftarrow{X}_g)^\dagger &= \overleftarrow{X}_{\bar{g}}. \end{aligned} \quad (9)$$

The two types of X operators can be combined to form the *conjugation* operator:

$$\overleftarrow{\overleftarrow{X}}_g := \vec{X}_g \overleftarrow{X}_g = \overleftarrow{X}_g \vec{X}_g. \quad (10)$$

The group-based Z -type operators are labelled by ir-

reps Γ and act as generalized phase gates:

$$Z_{\Gamma} = \sum_{g \in G} \Gamma(g) \otimes |g\rangle\langle g| \Leftrightarrow Z_{\Gamma} |g\rangle = \Gamma(g) \otimes |g\rangle, \quad (11)$$

$$Z_{\Gamma}^\dagger = \sum_{g \in G} \Gamma(\bar{g}) \otimes |g\rangle\langle g| \Leftrightarrow Z_{\Gamma}^\dagger |g\rangle = \Gamma(\bar{g}) \otimes |g\rangle. \quad (12)$$

These are matrix product operators (MPOs), with the irrep matrices acting in the virtual space. In graphical notation, we have

$$Z_{\Gamma} = \text{---} \boxed{Z_{\Gamma}} \text{---} = \sum_{g \in G} \Gamma(g) \otimes |g\rangle\langle g|. \quad (13)$$

In order to reduce these MPOs to operators acting solely on the physical degrees of freedom, we must contract over the virtual space. We can either do this by tracing over the space – equivalent to imposing periodic boundary conditions – which is denoted by

$$\text{Tr} \left[\prod_{i=1}^n Z_{\Gamma}^{(i)} \right] := \sum_{g_i \in G} \text{Tr} [\Gamma(g_1 \cdots g_n)] |g_1, \dots, g_n\rangle\langle g_1, \dots, g_n| = \text{---} \boxed{Z_{\Gamma}} \text{---} \cdots \text{---} \boxed{Z_{\Gamma}} \text{---} \text{---},$$

or by choosing a particular matrix element of Γ , denoted by

$$\left[\prod_{i=1}^n Z_{\Gamma}^{(i)} \right]_{\alpha\beta} := \sum_{g_i \in G} \langle \alpha | \Gamma(g_1 \cdots g_n) | \beta \rangle |g_1, \dots, g_n\rangle\langle g_1, \dots, g_n| = \text{---} \boxed{Z_{\Gamma}} \text{---} \cdots \text{---} \boxed{Z_{\Gamma}} \text{---} \text{---},$$

where the product is taken in ascending order of indices (clockwise along the loop of virtual links), $\text{---} \blacksquare \text{---} = \langle \alpha |$, and $\text{---} \blacksquare \text{---} = | \beta \rangle$. Here, $|\alpha\rangle$ and $|\beta\rangle$ are states in the virtual

Hilbert space of dimension d_{Γ} , not to be confused with physical states $|g\rangle$. Notice that Z_{Γ} is an operator acting on the physical degrees of freedom, while $\Gamma(g)$ is simply

a matrix.

Sums over Z -type operators labelled by different Γ can be equivalently written as projectors, and this notation is sometimes used in other works. To go from the Pauli picture to the projector picture, one uses the delta function on the group, given by

$$\delta_{g,h}^G = \sum_{\Gamma \in \text{Rep}(G)} \frac{d_\Gamma}{|G|} \text{Tr}[\Gamma(\bar{g}h)], \quad (14)$$

where the sum is over the irreps Γ of G , and where d_Γ is the dimension of the Γ irrep. For example, a sum over irreps of single-site $\text{Tr}[Z_\Gamma]$ operators projects onto the identity state:

$$\sum_{\Gamma \in \text{Rep}(G)} \frac{d_\Gamma}{|G|} \text{Tr}[Z_\Gamma] = |e\rangle\langle e|, \quad (15)$$

and the generalized two-site Ising interaction projects onto the subspace with spins aligned:

$$\sum_{\Gamma} \frac{d_\Gamma}{|G|} \text{Tr} \left[Z_\Gamma^{(1)\dagger} \cdot Z_\Gamma^{(2)} \right] = \sum_g |g, g\rangle\langle g, g|, \quad (16)$$

where the ‘ \cdot ’ denotes multiplication in the virtual space.

Group-based Pauli operators have more subtle commutation relations than their qudit counterparts due to non-commutativity of operators on the virtual space; the order in which the virtual space matrices are contracted matters. This comes from the fact that $\Gamma(g_i)$ and $\Gamma(g_j)$ do not in general commute. Indeed, for any non-abelian group, there will exist some irrep Γ with dimension $d_\Gamma > 1$ and some g_i, g_j such that $\Gamma(g_i)$ and $\Gamma(g_j)$ do not commute. This is important to keep in mind when working with the operators algebraically, but becomes more obvious when working diagrammatically.

The X operators commute with one another when they act from different sides or on different sites, and follow the commutation rules of the group otherwise:

$$\begin{aligned} [\vec{X}_g^{(i)}, \overleftarrow{X}_h^{(j)}] &= 0 \\ [\vec{X}_g^{(i)}, \vec{X}_h^{(j)}] &\propto [\overleftarrow{X}_g^{(i)}, \overleftarrow{X}_h^{(j)}] \propto \delta_{i,j}. \end{aligned} \quad (17)$$

The commutation relations between X and Z operators on the same site are given by

$$\begin{aligned} \vec{X}_g Z_\Gamma &= \Gamma(g) \cdot Z_\Gamma \vec{X}_g, \\ \overleftarrow{X}_g Z_\Gamma &= Z_\Gamma \cdot \Gamma(g) \overleftarrow{X}_g. \end{aligned} \quad (18)$$

Notice that these operators fail to commute up to a matrix $\Gamma(\bar{g})$ or $\Gamma(g)$ acting on the virtual space, rather than up to a phase as is the case for qubits.

In addition to the computational basis labelled by the group elements, there exists a dual basis labeled by matrix elements of irreps. These states are defined to be

$$|\Gamma_{\alpha\beta}\rangle := \sqrt{\frac{d_\Gamma}{|G|}} \sum_{g \in G} [\Gamma(g)]_{\alpha\beta} |g\rangle. \quad (19)$$

It can be shown using the Grand Orthogonality Theorem of group representations [138] that these states form a complete basis for $\mathbb{C}[G]$. In the qubit case $G = \mathbb{Z}_2$, these states indeed reduce to the states $|+\rangle$ and $|-\rangle$, corresponding to Γ being the trivial irrep and sign irrep, respectively³.

It is also sometimes convenient to think of the set of character states

$$|\Gamma\rangle := \sqrt{\frac{d_\Gamma}{|G|}} \sum_{g \in G} \text{Tr}[\Gamma(g)] |g\rangle, \quad (20)$$

which span a subspace of $\mathbb{C}[G]$.

III. G CLUSTER STATE

The family of cluster states based on finite groups was introduced by Brell in Ref. 55. Brell set out to generalize cluster states to a group-valued Hilbert space, analogous to the relationship between the toric code and Kitaev’s quantum double. However, the cluster states presented an additional hurdle not present in the toric code: stabilizers which mixed X and Z operators. To use the language of quantum error correction, the toric code is a CSS stabilizer code \mathbb{F}_2^{56} , and the ZXZ cluster state is not.

The difficulty of working with mixed stabilizers arises from the fact that there is no natural isomorphism between group elements and irreducible representations of a non-abelian group. The qubit cluster state is stabilized by ZXZ operators. These would naively generalize to stabilizers like $Z_\Gamma^\dagger \vec{X}_g Z_\Gamma$, but it is not obvious which g should be paired with which Γ in a given term. For \mathbb{Z}_2 , for example, the natural pairing between the computational and dual basis is $|0\rangle \leftrightarrow |+\rangle$ and $|1\rangle \leftrightarrow |-\rangle$. When G is non-abelian, however, there is no natural pairing, in part because there are more group elements than irreps.

Brell circumvented this issue by introducing the CSS cluster state (see Fig. 1), which can be generalized from qubits to group-valued Hilbert spaces. The CSS cluster state is generated by acting on the 1d cluster state with a Hadamard gate on the even sites. The CSS cluster state has new stabilizers, which are also obtained through the action of the Hadamard gate. The stabilizers beginning on odd sites are transformed from ZXZ to ZZZ , and the stabilizers beginning on even sites from ZXZ to XXX . Since the stabilizers now consist of only a single type of Pauli operator, the new state is a CSS stabilizer state.

The CSS cluster state can also be prepared by acting with a circuit of CX gates on the product state $|+\rangle|0\rangle|+\rangle \dots$. This method of state preparation generalizes to the finite-group case. Noticing that, for $G = \mathbb{Z}_2$,

³ \mathbb{Z}_2 is abelian and therefore only has one dimensional irreps, making the use of indices $\alpha\beta$ unnecessary in this case.

$$\begin{array}{c} \boxed{\overleftarrow{X}_g} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \boxed{\overleftarrow{X}_g} \end{array} \quad (26d)$$

$$\begin{array}{c} \boxed{Z_\Gamma} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \boxed{Z_\Gamma} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \boxed{Z_\Gamma} \end{array} \quad (26e)$$

$$\begin{array}{c} \boxed{Z_\Gamma} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \boxed{Z_\Gamma} \end{array} \quad (26f)$$

B. Stabilizer Hamiltonian and Symmetries

We can construct a Hamiltonian with ground states $|\mathcal{C}\rangle$ as a sum of commuting stabilizers:

$$H_{\mathcal{C}} = -\frac{1}{|G|} \sum_{i \text{ odd}} \left(\sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_\Gamma^{(i)} \cdot Z_\Gamma^{(i+1)} \cdot Z_\Gamma^{(i+2)} \right] d_\Gamma + \sum_{g \in G} \overleftarrow{X}_g^{(i+1)} \overrightarrow{X}_g^{(i+2)} \overrightarrow{X}_g^{(i+3)} \right), \quad (27)$$

where the sum over $\Gamma \in \text{Rep}(G)$ denotes a sum over the simple objects of $\text{Rep}(G)$, namely the irreps Γ of G . We call this a (generalized) stabilizer Hamiltonian because the group-based Pauli terms all commute with one another, and the ground state space is the joint eigenspace of all operators with maximum eigenvalue. For the X -type terms, this maximum eigenvalue is $1/|G|$, and for the Z -type terms, it is $d_\Gamma^2/|G|$. We choose to normalize to these values rather than to 1 because the Hamiltonian also becomes a commuting projector Hamiltonian when written with this normalization.

The Hamiltonian Eq. (27) has *four* independent families of global symmetries, all of which are respected by the ground state $|\mathcal{C}\rangle$. There are two G symmetries, given by tensor products of X -type operators, a $\text{Rep}(G)$ symmetry comprised of Z -type operators which acts as an MPO, and an $\text{Inn}(G)$ symmetry:

$$G_R : \overleftarrow{A}_g := \prod_{i \text{ odd}} \overleftarrow{X}_g^{(i)}, \quad (28a)$$

$$G_L : \overrightarrow{A}_g := \prod_{i \text{ odd}} \overrightarrow{X}_g^{(i)} \overrightarrow{X}_g^{(i+1)}, \quad (28b)$$

$$\text{Rep}(G) : \hat{B}_\Gamma := \text{Tr} \left[\prod_{i \text{ even}} Z_\Gamma^{(i)} \right], \quad (28c)$$

$$\text{Inn}(G) : \hat{C}_g := \prod_{i \text{ even}} \left(\sum_{g'} \overleftarrow{X}_{g'g\bar{g}}^{(i)} \delta_{g'}^{(i+1)} \overrightarrow{X}_{g'g\bar{g}}^{(i+2)} \right), \quad (28d)$$

MPS	MPO
$\bullet = \sum_{g \in G} g\rangle\langle g \otimes g\rangle$	$\boxed{\overleftarrow{X}_g} = \sum_{h \in G} \mathbb{1} \otimes gh\rangle\langle h $
$\circ = \sum_{g \in G} \overrightarrow{X}_g \otimes g\rangle$	$\boxed{\overrightarrow{X}_g} = \sum_{h \in G} \mathbb{1} \otimes hg\rangle\langle h $
$\bullet = \sum_{g \in G} g^n\rangle\langle g^n \otimes g\rangle$	$\boxed{Z_\Gamma} = \sum_{g \in G} \Gamma(g) \otimes g\rangle\langle g $

TABLE II. Summary of matrix product state and operator tensors used throughout this work.

where $\text{Inn}(G)$ is the inner automorphism group of G . $\text{Inn}(G) \cong G/Z(G)$, where $Z(G)$ is the center of G , the subgroup of elements in G which commute with every other element of G .

For abelian groups, the two G symmetries collapse to one, and the $\text{Inn}(G)$ symmetry becomes trivial, yielding only two symmetries, $G \times \text{Rep}(G)$, that protect the SPT. Accordingly, we have determined analytically and numerically that $G_R \times \text{Rep}(G)$ is the minimal subgroup which protects the SPT in the non-abelian case as well, so we focus on these symmetries for the remainder of the text. We discuss the additional symmetries in Appendix B and reserve a full discussion of alternative avenues to protection to future work.

Notice that \overleftarrow{A}_g and \hat{B}_Γ commute with one another, as they are supported on different sublattices. The \overleftarrow{A} operators form a representation of the group G , as

$$\overleftarrow{A}_g \overleftarrow{A}_h = \overleftarrow{A}_{gh}. \quad (29)$$

Diagrammatically, \hat{B}_Γ can be written

$$\hat{B}_\Gamma = \boxed{Z_\Gamma} \text{---} \dots \text{---} \boxed{Z_\Gamma}.$$

When we act on an open chain, we will instead use the open form of the symmetry operator:

$$\hat{B}_{\Gamma_{\alpha\beta}} := \left[\prod_{i \text{ even}} Z_\Gamma^{(i)} \right]_{\alpha\beta} = \text{---} \boxed{Z_\Gamma} \text{---} \dots \text{---} \boxed{Z_\Gamma} \text{---} \text{---} \quad (30)$$

The Z -type operators do not generically multiply according to the rules of a group. As derived in Appendix D, the \hat{B} operators multiply according to the rules of the fusion category $\text{Rep}(G)$, the category of representations of G :

$$\hat{B}_{\Gamma_i} \hat{B}_{\Gamma_j} = \sum_k N_{\Gamma_i, \Gamma_j}^{\Gamma_k} \hat{B}_{\Gamma_k}, \quad (31)$$

where the $N_{\Gamma_i, \Gamma_j}^{\Gamma_k}$ encode the fusion rules of $\text{Rep}(G)$. We say that the \hat{B} operators realize a *fusion category* symmetry [58, 60]. Such symmetries have also occasionally been called *algebraic* symmetries in the literature.

Fusion category symmetries are also sometimes called non-invertible symmetries [141, 142], especially in the high energy context. This applies to our symmetry in that Z_Γ does not in general have an inverse. This is a consequence of the fact that for each simple object Γ in a fusion category, there must exist a dual simple object $\bar{\Gamma}$ such that $\Gamma \times \bar{\Gamma} = \mathbf{1} + \dots$. That is, each object must have a dual which fuses with it to yield the trivial object, but this need not be the only fusion channel.

In the case of abelian G , the algebraic structure in Eq. (31) reduces to a group structure because $\text{Rep}(G) \cong G$ when G is abelian. This is to say that the character group is isomorphic to the group when G is abelian. We will focus on the case where G is non-abelian, so the Z -type symmetry is not group-like.

C. On $G \times \text{Rep}(G)$

Together, the \overleftarrow{A} and \hat{B} operators realize a $G \times \text{Rep}(G)$ fusion category symmetry. Simple objects in this category can be labeled by the pair (g, Γ) and satisfy the fusion rules

$$(g, \Gamma_i) \otimes (h, \Gamma_j) = \bigoplus_k N_{\Gamma_i, \Gamma_j}^{\Gamma_k} (gh, \Gamma_k). \quad (32)$$

Accordingly, the symmetry operators $\overleftarrow{A}_g \hat{B}_\Gamma$ multiply as

$$\left(\overleftarrow{A}_g \hat{B}_{\Gamma_i} \right) \left(\overleftarrow{A}_h \hat{B}_{\Gamma_j} \right) = \sum_k N_{\Gamma_i, \Gamma_j}^{\Gamma_k} \overleftarrow{A}_{gh} \hat{B}_{\Gamma_k}. \quad (33)$$

The symmetry category is a direct product of G and $\text{Rep}(G)$ because the operators realizing the G and $\text{Rep}(G)$ symmetries commute and therefore act entirely independently.

Interestingly, $G \times \text{Rep}(G)$ has non-commutative fusion operation \times , as $gh \neq hg$ in general. This is not to be confused with the statement that fusion rules are said to be non-abelian when $N_{\Gamma_i, \Gamma_j}^{\Gamma_k}$ is nonzero for multiple Γ_k , given fixed Γ_i and Γ_j . The latter comes from the fact that the fusion rules of quantum double anyons derived from non-abelian groups satisfy such a condition. However, the fusion operation of non-abelian anyons is still commutative in the sense that $a \times b = b \times a$ for two non-abelian anyons a and b . Indeed, it is not possible to construct a physical theory of anyons based on the fusion rules given in Eq. (32). This is because commutativity of the fusion operation \times is necessary for the braiding operation to be well-defined [50].

The fact that the fusion operation of $G \times \text{Rep}(G)$ is non-commutative also implies that it is not the fusion algebra of the irreps of any finite group: $G \times \text{Rep}(G) \neq \text{Rep}(G')$ for any G' . This is because the fusion of irreps of any finite group is always commutative – though, of course, not always abelian in the single-fusion-channel sense. However, every non-anomalous fusion category is isomorphic to the representation category

of some semisimple Hopf algebra [31]. This implies that there exists some semisimple Hopf algebra H such that $G \times \text{Rep}(G) \cong \text{Rep}(H)$. If one were to construct commuting projector Hamiltonians for phases with $G \times \text{Rep}(G)$ symmetry following the prescription of [85], this is the Hopf algebra one would use.

IV. DISTINCTNESS FROM THE SYMMETRIC PRODUCT STATE

In this section, we will first discuss what it means for a state to be protected by a fusion category symmetry and understand what notion of triviality and non-triviality exists for such states. We will then construct a Kramers-Wannier duality which maps the $G \times \text{Rep}(G)$ -symmetric phases in which we are interested to $G \times G$ spontaneous symmetry breaking phases which can be more easily understood. This duality allows us to conclude that the G cluster state is non-trivial, in that it is not adiabatically connected to the symmetric $G \times \text{Rep}(G)$ product state via any symmetric path of Hamiltonians, and that this order is robust to weak symmetric local perturbations.

A. Definition of SPT with fusion-category symmetry

Given a fusion category \mathcal{A} , we define an \mathcal{A} -SPT state as the unique ground state of a gapped Hamiltonian which respects the symmetry \mathcal{A} under periodic boundary conditions. Two \mathcal{A} -symmetric states belong to distinct phases if they cannot be deformed into one another by a finite time evolution of a symmetric local Hamiltonian [58]. $|\psi_0\rangle := |\mathbf{1}, e, \mathbf{1}, e, \dots\rangle$ is a product state that respects the $G \times \text{Rep}(G)$ symmetry, so we can use it as a representative symmetric product state. Since the state $|\mathcal{C}\rangle$ is also a $G \times \text{Rep}(G)$ -symmetric state, our goal is to show that it belongs to a distinct phase from the product state $|\psi_0\rangle$.

It is important to emphasize here and throughout the paper that we will be intentionally using the phrase (symmetric) *product* state in lieu of the *trivial* SPT state. To clarify our nomenclature, there are two possible definitions one can use to define a trivial SPT phase

1. It is the identity element of the group of SPT phases with stacking as a group operation
2. It is the phase to which the symmetric product state belongs

In the case of invertible symmetries, these two definitions are equivalent. However, SPTs protected by non-invertible symmetries cannot be stacked, and therefore only form a set rather than a group. To see this, it is helpful to review how to stack SPT phases when the symmetry is unitary. Consider two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 each hosting a G -symmetric SPT phase $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. To stack the two phases, we may consider the

tensor product state $|\psi_1\rangle \otimes |\psi_2\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and restrict the full symmetry $G \times G$ to the diagonal subgroup G which acts on both copies at the same time. Then, given a choice of a local symmetry action, the trivial SPT phase then contains all wavefunctions $|\psi_0\rangle$ such that $|\psi\rangle \otimes |\psi_0\rangle$ is in the same phase as $|\psi\rangle$ as G -symmetric phases. In contrast, a general fusion-category symmetry $\mathcal{A} \times \mathcal{A}$ may not contain \mathcal{A} as a diagonal subcategory. Thus there is generally no notion of stacking, and therefore the trivial SPT phase in the first sense cannot be defined.

On the other hand, having chosen a tensor product Hilbert space and a representation for the symmetry, the symmetric product state lives in one and only one of these SPT phases. To conclude, there is a “special” SPT phase containing the symmetric product state. Thus, the trivial SPT phase in the second sense can be defined if a decomposition of the Hilbert space is chosen.

Relatedly, although such SPT phases can be unambiguously called *short-range entangled* phases since they can be deformed to a product state by explicitly breaking the symmetry, it is ambiguous to define whether an SPT state is *invertible*. In the first sense, invertibility is not well defined due to the lack of a group structure. In

the second sense, it is possible the unitary can be used to permute between different SPT phases. Since every unitary has an inverse, these SPTs can be thought of as invertible in the second sense if the symmetric product state is chosen as a base point. However, if we do not declare the symmetric product state as a special phase, then this unitary only gives rise to a group action on the set of SPT phases (i.e. a *torsor* [60]).

B. Duality argument

We construct a Kramers-Wannier duality [143] which associates the G cluster state $|\mathcal{C}\rangle$ and the $G \times \text{Rep}(G)$ -symmetric product state $|\psi_0\rangle$ to two different $G \times G$ symmetry-breaking states. States with different patterns of spontaneous symmetry breaking belong to different phases, implying through the duality that $|\mathcal{C}\rangle$ and $|\psi_0\rangle$ also belong to different phases.

We begin by constructing two models which spontaneously break $G \times G$ symmetry:

$$\begin{aligned} H_{SSB1} &= -\frac{1}{|G|} \sum_{i \text{ odd}} \left(\sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_{\Gamma}^{(i+1)} \cdot Z_{\Gamma}^{\dagger(i+3)} \right] d_{\Gamma} + \sum_{g \in G} \vec{X}_g^{(i)} \right) = -\sum_{i \text{ odd}} \left(\delta_{g_{i+1}, g_{i+3}}^G + \frac{1}{|G|} \sum_{g \in G} \vec{X}_g^{(i)} \right), \\ H_{SSB2} &= -\frac{1}{|G|} \sum_{i \text{ odd}} \left(\sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_{\Gamma}^{\dagger(i)} \cdot Z_{\Gamma}^{(i+1)} \cdot Z_{\Gamma}^{\dagger(i+3)} \cdot Z_{\Gamma}^{(i+2)} \right] d_{\Gamma} + \sum_{g \in G} \vec{X}_g^{(i)} \vec{X}_g^{(i+1)} \right). \end{aligned} \quad (34)$$

Both of these Hamiltonians have a family of symmetries on each sublattice:

$$G_R^{\text{odd}} : \prod_{i \text{ odd}} \vec{X}_g^{(i)}, \quad G_R^{\text{even}} : \prod_{i \text{ even}} \vec{X}_g^{(i)}. \quad (35)$$

These symmetries act independently, so the total symmetry group is $G_R^{\text{odd}} \times G_R^{\text{even}}$. Symmetry is spontaneously broken in both models. The ground-state subspaces are given by

$$\begin{aligned} H_{SSB1} &: \{ |\mathbf{1}, g, \mathbf{1}, g, \dots\rangle : g \in G \}, \\ H_{SSB2} &: \left\{ \sum_{\{g_i\} \in G} |g_1, g_1 g, g_2, g_2 g, \dots\rangle : g \in G \right\}. \end{aligned} \quad (36)$$

In *SSB1*, the unbroken symmetry is G_R^{odd} . In *SSB2* the unbroken symmetry for the state labeled by g is $G_g := \{ \vec{X}_h^{(i)} \vec{X}_{ghg}^{(i+1)} : h \in G \}$. In each case, the symmetry is broken from $G \times G \rightarrow G$, but with different unbroken subgroups. Thus, the two families of ground states realize distinct gapped phases under the $G_R^{\text{odd}} \times G_R^{\text{even}}$ symmetry.

Notice that *SSB1* and *SSB2* are related by a controlled multiplication from the odd sublattice to the even. In

particular, let

$$U_{\text{SSB}} := \prod_{i \text{ odd}} C \vec{X}^{(i, i+1)}, \quad (37)$$

then we see that

$$\begin{aligned} U_{\text{SSB}} |\mathbf{1}, g, \mathbf{1}, g, \dots\rangle &= \sum_{\{g_i\} \in G} |g_1, g_1 g, g_2, g_2 g, \dots\rangle, \\ U_{\text{SSB}} H_{SSB1} (U_{\text{SSB}})^{\dagger} &= H_{SSB2}. \end{aligned} \quad (38)$$

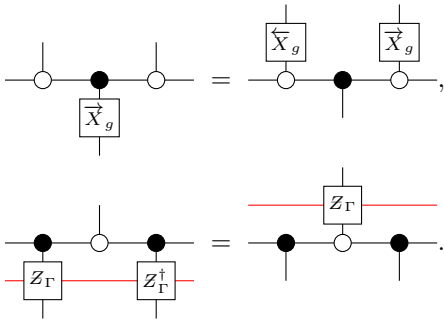
We now want to apply Kramers-Wannier duality to the even sites, taking the $G \times G$ symmetry to $G \times \text{Rep}(G)$. This duality can also be thought of as gauging the G_R^{even} subgroup of $G_R^{\text{odd}} \times G_R^{\text{even}}$ in the sense that the original theory is pure matter and the KW dual theory is pure gauge. Gauging of finite subgroups was discussed previously in Ref. 144.

It was shown in Ref. 145 (see Appendix E for a brief review) that the KW duality can be implemented via the cluster state MPS by flipping the physical leg of the odd-site tensors. Using the MPS for the G cluster state, we

Symmetry	States		
$G \times G$	<i>SSB1</i>	<i>SSB2</i>	Phase
	$\{ \mathbf{1}, g, \mathbf{1}, g, \mathbf{1}, g, \dots\rangle : g \in G\}$	$\{\sum_{\{g_i\} \in G} g_1, g_1 g_2, g_2, g_2 g_3, g_3, g_3 g_4, \dots\rangle : g \in G\}$	Ground State Manifold
	$\text{Tr}[Z_\Gamma^\dagger \cdot \mathbf{1} \cdot Z_\Gamma]$ \vec{X}_g $\text{Tr}[Z_\Gamma^\dagger \cdot \mathbf{1} \cdot Z_\Gamma]$ \vec{X}_g	$\text{Tr}[Z_\Gamma^\dagger \cdot Z_\Gamma \cdot Z_\Gamma \cdot Z_\Gamma^\dagger]$ $\vec{X}_g \vec{X}_g$ $\text{Tr}[Z_\Gamma^\dagger \cdot Z_\Gamma \cdot Z_\Gamma \cdot Z_\Gamma^\dagger]$ $\vec{X}_g \vec{X}_g$	Stabilizers
	$\sum_{\Gamma \neq \mathbf{1}} \frac{d_\Gamma}{ G -1} \text{Tr} \left[Z_\Gamma^{\dagger(\text{even})} \cdot Z_\Gamma^{(\text{even}+2k)} \right]$	$\sum_{\Gamma \neq \mathbf{1}} \frac{d_\Gamma}{ G -1} \text{Tr} \left[Z_\Gamma^{\dagger(\text{odd})} \cdot Z_\Gamma^{(\text{even})} \cdot Z_\Gamma^{\dagger(\text{even}+2k)} \cdot Z_\Gamma^{(\text{odd}+2k)} \right]$	Order Parameter
	$\frac{1}{ G -1} \sum_{g \neq e} \left(\prod_{j=0}^k \vec{X}_g^{(\text{odd}+2j)} \right)$	$\frac{1}{ G -1} \sum_{g \neq e} \left(\prod_{j=0}^k \vec{X}_g^{(\text{odd}+2j)} \vec{X}_g^{(\text{even}+2j)} \right)$	Disorder Parameter
$G \times \text{Rep}(G)$	<i>Product State</i>	<i>G Cluster State (SPT)</i>	Phase
	$ \mathbf{1}, e, \mathbf{1}, e, \mathbf{1}, e, \dots\rangle$	$\sum_{\{g_i\} \in G} g_1, g_1 \bar{g}_2, g_2, g_2 \bar{g}_3, g_3, g_3 \bar{g}_4, \dots\rangle$	Unique Ground State
	$\text{Tr}[Z_\Gamma^\dagger]$ \vec{X}_g $\text{Tr}[Z_\Gamma^\dagger]$ \vec{X}_g	$\text{Tr}[Z_\Gamma^\dagger \cdot Z_\Gamma \cdot Z_\Gamma]$ $\vec{X}_g \vec{X}_g \vec{X}_g$ $\text{Tr}[Z_\Gamma^\dagger \cdot Z_\Gamma \cdot Z_\Gamma]$ $\vec{X}_g \vec{X}_g \vec{X}_g$	Stabilizers
	$\sum_{\Gamma \neq \mathbf{1}} \frac{d_\Gamma}{ G -1} \text{Tr} \left[\prod_{j=0}^k Z_\Gamma^{\dagger(\text{even}+2j)} \right]$	$\sum_{\Gamma \neq \mathbf{1}} \frac{d_\Gamma}{ G -1} \text{Tr} \left[Z_\Gamma^{\dagger(\text{odd})} \cdot \left(\prod_{j=0}^{k-1} Z_\Gamma^{(\text{even}+2j)} \right) \cdot Z_\Gamma^{(\text{odd}+2k)} \right]$	String Order Parameter
	$\frac{1}{ G -1} \sum_{g \neq e} \left(\prod_{j=0}^k \vec{X}_g^{(\text{odd}+2j)} \right)$	$\frac{1}{ G -1} \sum_{g \neq e} \vec{X}_g^{(\text{even})} \vec{X}_g^{(\text{odd})} \left(\prod_{j=1}^{k-1} \vec{X}_g^{(\text{even}+2j)} \vec{X}_g^{(\text{odd}+2j)} \right) \vec{X}_g^{(\text{even}+2k)}$	Disorder Parameter

TABLE III. A summary of the four quantum phases involved in the duality argument. The double arrow in the $ZZZZ$ stabilizers of SSB2 reflects the fact that the operators corresponding to the third and fourth sites appear in reverse order in the virtual space matrix product, see Eq. (34). The data associated with each phase can be derived from the data of any other phase using the quantum circuits and dualities laid out in Fig. 2.

find:



From this diagrammatic derivation, we read off the appropriate KW duality:

$$\begin{aligned} Z_\Gamma^{(i+1)} \cdot Z_\Gamma^{(i+3)\dagger} &\mapsto Z_\Gamma^{(i+1)}, \\ \vec{X}_g^{(i+1)} &\mapsto \vec{X}_g^{(i-1)} \vec{X}_g^{(i+1)}. \end{aligned} \quad (39)$$

This is a generalized domain wall duality. Applying the

duality to each of our SSB Hamiltonians, we find

$$\begin{aligned} H_{SSB1} &\mapsto - \sum_{i \text{ odd}} \sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_\Gamma^{\dagger(i+1)} \right] d_\Gamma + \sum_{g \in G} \vec{X}_g^{(i)}, \\ H_{SSB2} &\mapsto - \sum_{i \text{ odd}} \sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_\Gamma^{\dagger(i)} \cdot Z_\Gamma^{(i+1)} \cdot Z_\Gamma^{(i+2)} \right] d_\Gamma \\ &\quad + \sum_{g \in G} \vec{X}_g^{(i-1)} \vec{X}_g^{(i)} \vec{X}_g^{(i+1)} = H_C, \end{aligned} \quad (40)$$

Notice that H_{SSB2} maps to H_C under the KW duality, while H_{SSB1} maps to a Hamiltonian with unique ground state $|\psi_0\rangle := |\mathbf{1}, e, \mathbf{1}, e, \dots\rangle$, our reference $G \times \text{Rep}(G)$ -symmetric product state.

We can use this duality to prove that the G cluster state is not in the same phase as the symmetric product state as follows: because H_{SSB1} and H_{SSB2} exhibit different patterns of spontaneous symmetry breaking, they belong to different phases. We have just shown that these Hamiltonians are mapped under the KW duality into the product state Hamiltonian and G cluster state Hamiltonian, respectively. If there were a symmetric path con-

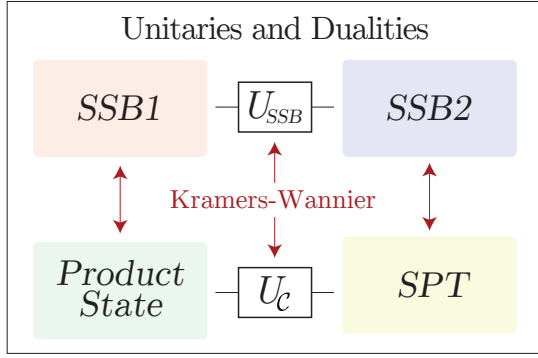


FIG. 2. A depiction of the KW duality and quantum circuits relating the phases tabulated in Table III. Note that the quantum circuits themselves – defined in Eq. (37) and Eq. (23) – are also related through the KW duality.

necting the G cluster state to the product state, then pushing this path through the KW duality would give a symmetric path connecting the two SSB states, which contradicts the fact that they belong to distinct phases. This implies that there does not exist a symmetric path connecting the cluster state to the product state, so that they must be in separate phases. We can conclude that $|\mathcal{C}\rangle$ belongs to a distinct fusion category SPT phase from the symmetric product state.

We can further argue that the fusion category SPT phase to which the G belongs is robust to weak symmetric local perturbations. Consider such a perturbation $\delta H_{G \times \text{Rep}(G)}$, where the subscript denotes the symmetry respected by the perturbation. Beginning with H_C , the perturbed Hamiltonian becomes

$$H'_C = H_C + \delta H_{G \times \text{Rep}(G)}. \quad (41)$$

We want to know whether H'_C describes the same phase as H_C , for any choice of $\delta H_{G \times \text{Rep}(G)}$, which is equivalent to saying that the perturbation doesn't close the gap. Because local symmetric operators are mapped to local symmetric operators through the KW duality [143], we can map H'_C to

$$H'_{SSB1} = H_{SSB1} + \delta H_{G \times G}, \quad (42)$$

where $H_{G \times G}$ is a local $G \times G$ -symmetric perturbation. We know that the SSB order of H_{SSB1} is robust to local symmetric perturbations, so we can conclude that the ground states of H_{SSB1} and H'_{SSB1} lie in the same phase. From this we know that the ground states of the Hamiltonians they map into under the KW duality – namely H_C and H'_C – lie in the same phase. This means that the SPT order in $|\mathcal{C}\rangle$ is robust; in particular, its ground state degeneracy – to be discussed in the following section – is robust to local symmetric perturbations. We verify robustness of ground-state degeneracy numerically for the simplest non-abelian case in Sec. VI A.

V. SIGNATURES OF SPT ORDER

In this section, we discuss how characteristic signatures of SPT order generalize to the context of fusion category symmetry. For a review of the qubit results which are generalized here, see Appendix A.

A. Edge Modes and Ground State Degeneracy

1d SPTs on open chains with on-site group symmetry action support edge modes that are robust to local symmetric perturbations. These edge modes are responsible for the ground state degeneracy of these phases. In the bosonic case, the edge modes transform as a projective representation of the global symmetry G and can be identified with elements of the second cohomology group $\mathcal{H}^2(G, U(1))$ [52, 114, 120], which in turn provides a classification of 1d bosonic G -SPTs.

We can detect the presence of edge modes by acting on the state $|\mathcal{C}\rangle$ with the global symmetries \hat{A}_g and \hat{B}_Γ . We find that the action of the global symmetries pulls through to the virtual level and reduces to operators acting only on the edge Hilbert spaces \mathcal{H}_L and \mathcal{H}_R :

$$\begin{aligned} \hat{A}_g |\mathcal{C}\rangle &= \begin{array}{c} \boxed{\hat{X}_g} \quad \boxed{\hat{X}_g} \quad \dots \quad \boxed{\hat{X}_g} \quad \boxed{\hat{X}_g} \\ \bullet \quad \circ \quad \bullet \quad \dots \quad \bullet \quad \circ \quad \bullet \end{array} \\ &= \begin{array}{c} \boxed{\hat{X}_g^\dagger} \quad \bullet \quad \circ \quad \bullet \quad \dots \quad \bullet \quad \circ \quad \bullet \quad \boxed{\hat{X}_g} \end{array}, \\ \hat{B}_\Gamma |\mathcal{C}\rangle &= \begin{array}{c} \boxed{Z_\Gamma} \quad \dots \quad \boxed{Z_\Gamma} \\ \bullet \quad \circ \quad \bullet \quad \dots \quad \bullet \quad \circ \quad \bullet \end{array} \\ &= \begin{array}{c} \boxed{Z_\Gamma} \quad \bullet \quad \circ \quad \bullet \quad \dots \quad \bullet \quad \circ \quad \bullet \quad \boxed{Z_\Gamma^\dagger} \end{array}, \end{aligned}$$

where $\blacksquare = \langle \psi_L | \in \mathcal{H}_L^*$ and $\blacksquare = |\psi_R\rangle \in \mathcal{H}_R$. From this diagrammatic derivation, we can read off the edge modes:

$$G : \hat{X}_g^{(L)} \hat{X}_g^{(R)}, \quad \text{Rep}(G) : \left[Z_\Gamma^{(L)} Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta}, \quad (43)$$

which act on $\mathcal{H}_L \otimes \mathcal{H}_R$. We have written $\hat{X}_g^{(L)}$ rather than $\hat{X}_g^{(L)\dagger}$ because $\langle \psi_L | \hat{X}_g^{(L)\dagger} \mapsto \hat{X}_g^{(L)} | \psi_L \rangle$. It is also possible to derive the existence of edge modes without knowing the form of the ground state, using only the stabilizer Hamiltonian Eq. (27) (see Appendix F).

Notice that the edge modes acting on each edge do not commute with one another:

$$\begin{aligned} \hat{X}_g^{(L)} \left[Z_\Gamma^{(L)} Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} &= \left[Z_\Gamma^{(L)} \Gamma(g) Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} \hat{X}_g^{(L)}, \\ \hat{X}_g^{(R)} \left[Z_\Gamma^{(L)} Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} &= \left[Z_\Gamma^{(L)} \Gamma(\bar{g}) Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} \hat{X}_g^{(R)}. \end{aligned} \quad (44)$$

In the qubit case, the edge modes fail to commute up to a phase of ± 1 , reflecting the fact that the edge modes generate the Pauli matrices, a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Here, we find that the edge modes for group-valued qudits fail to commute *up to a matrix* $\Gamma(g)$ or $\Gamma(\bar{g})$. However, the total edge operators – the product of the left and right operators – still commute,

$$\overleftarrow{X}_g^{(L)} \overleftarrow{X}_g^{(R)} \left[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} = \left[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} \overleftarrow{X}_g^{(L)} \overleftarrow{X}_g^{(R)}, \quad (45)$$

because the extra factors of $\Gamma(g)$ and $\Gamma(\bar{g})$ meet between the Z operators and multiply to the identity.

These edge modes are also manifest in the wavefunctions of the ground states. On an open chain, the ground state manifold is spanned by $|G|^2$ basis states given by

$$|g_1\rangle \left(\mathcal{N} \sum_{g_2} \cdots \sum_{g_{N-1}} |g_1 \bar{g}_2\rangle |g_2\rangle \cdots |g_{N-1} \bar{g}_N\rangle \right) |g_N\rangle, \quad (46)$$

for $g_1, g_N \in G$. We can denote the state in Eq. (46) by $|g_1^{(L)}, g_N^{(R)}\rangle$. Notice that these states, expressed as matrix product states, exhibit the same bulk structure subject to different boundary conditions.

Let us show that the symmetry action of \overleftarrow{A}_g and $\hat{B}_{\Gamma_{\alpha\beta}}$ on the edge modes given in Eq. (43) can map between all basis states. This can be seen by performing a basis transformation by the unitary $C \overleftarrow{X}^{(L,R)}$, which maps

$$\overleftarrow{X}_g^{(L)} \overleftarrow{X}_g^{(R)} \mapsto \overleftarrow{X}_g^{(L)}, \quad \left[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta} \mapsto \left[Z_\Gamma^{\dagger(R)} \right]_{\alpha\beta}. \quad (47)$$

In this new basis, we may start with the basis state $|e^{(L)}, \mathbf{1}^{(R)}\rangle = \sum_{g \in G} |e^{(L)}, g^{(R)}\rangle$. Then by acting with

each symmetry, we find

$$\overleftarrow{A}_g \hat{B}_{\Gamma_{\alpha\beta}} |e^{(L)}, \mathbf{1}^{(R)}\rangle = |g^{(L)}, \Gamma_{\alpha\beta}^{(R)}\rangle. \quad (48)$$

Since $|g^{(L)}, \Gamma_{\alpha\beta}^{(R)}\rangle$ forms an orthonormal basis for $\mathbb{C}[G] \otimes \mathbb{C}[G]$, we have $|G|^2$ ground states.

B. String Order Parameters

A large class of SPTs admit a *non-local* or *string order parameter* which can effectively detect SPT order [121–125] (see Ref. [146] for an exception). We can derive two families of string order parameters for the G cluster state by mapping the order and disorder parameters in SSB2 through the KW duality. Because these parameters distinguish SSB1 from SSB2 on one side of the duality, we expect that they will be able to distinguish $|\psi_0\rangle$ from $|\mathcal{C}\rangle$ on the other side.

The Z -type string order parameter can be derived by mapping the order parameter from SSB2. It consists of a string of Z_Γ on even sites, with a Z_Γ^\dagger and Z_Γ on the odd sites preceding and following the string, respectively:

$$\hat{\mathcal{S}}_Z^{(i,k)} = \sum_{\Gamma \neq 1} \frac{d_\Gamma}{|G|-1} \text{Tr} \left[Z_\Gamma^{\dagger(i)} \cdot \left(\prod_{j=0}^{k-1} Z_\Gamma^{(i+1+2j)} \right) \cdot Z_\Gamma^{(i+2k)} \right], \quad (49)$$

where the superscript indexes the left endpoint i and the length $2k+1$ of the string operator. Notice also that $\hat{\mathcal{S}}_Z^{(i,k)}$ is Hermitian, so that its expectation value will always be real.

Diagrammatically, $\hat{\mathcal{S}}_Z^{(i,k)}$ acts on the G cluster state as

$$\hat{\mathcal{S}}_Z^{(i,k)} |\mathcal{C}\rangle = \sum_{\Gamma \neq 1} \frac{d_\Gamma}{|G|-1} \left(\text{Diagram of string operator} \right) |\mathcal{C}\rangle = \left(\sum_{\Gamma \neq 1} \frac{d_\Gamma^2}{|G|-1} \right) |\mathcal{C}\rangle = |\mathcal{C}\rangle, \quad (50)$$

as $\sum_\Gamma d_\Gamma^2 = |G|$ so that $\sum_{\Gamma \neq 1} d_\Gamma^2 = |G|-1$. We can then conclude that $\langle \mathcal{C} | \hat{\mathcal{S}}_Z | \mathcal{C} \rangle = 1$, as expected for a (string) order parameter.

We will now show that the expectation value of $\hat{\mathcal{S}}_Z^{(i,k)}$ in the $G \times \text{Rep}(G)$ -symmetric product state $|\psi_0\rangle$ is zero, so that $\hat{\mathcal{S}}_Z^{(i,k)}$ can be used to detect the SPT order in $|\mathcal{C}\rangle$. First, recall that the summation of the trace of a Z operator over all irreps projects the argument to the identity element (see Eq. (14)). We can use this to relate

$\hat{\mathcal{S}}_Z^{(i,k)}$ to the delta function on G :

$$\hat{\mathcal{S}}_Z^{(i,k)} = \frac{|G|}{|G|-1} \delta_{e, \bar{g}_i (\prod_{j=0}^{k-1} g_{i+1+2j}) g_{i+2k}} - \frac{1}{|G|-1}. \quad (51)$$

It directly follows that indeed $\langle \psi_0 | \hat{\mathcal{S}}_Z | \psi_0 \rangle = 0$, the details of which can be found in Appendix G.

The X -type string order parameter can be derived by mapping the *disorder* parameter from SSB2 through the KW duality. It consists of a string of \overleftarrow{X}_g on odd sites and \overrightarrow{X}_g on even sites, with an \overleftarrow{X}_g and $\overrightarrow{X}_g \overleftarrow{X}_g$ on the sites preceding and following the string, respectively:

$$\hat{\mathcal{S}}_X^{(i,k)} = \frac{1}{|G|-1} \sum_{g \neq e} \overleftarrow{X}_g^{(i)} \overrightarrow{X}_g^{(i+1)} \left(\prod_{j=1}^{k-1} \overleftarrow{X}_g^{(i+2j)} \overrightarrow{X}_g^{(i+1+2j)} \right) \overrightarrow{X}_g^{(i+2k)}, \quad (52)$$

where the superscript indexes the left endpoint i and the length $2k+1$ of the string operator. Note that $\hat{\mathcal{S}}_X^{(i,k)}$ begins on an even site.

Because $|\mathcal{C}\rangle$ is a $+1$ eigenstate of each term in the summand and there are $|G|-1$ terms, we have $\langle \mathcal{C} | \hat{\mathcal{S}}_X | \mathcal{C} \rangle = 1$. Furthermore, it is straightforward to see that each term in the summand has expectation value zero in $|\psi_0\rangle$ because factors of $\langle e|g\rangle = \delta_{e,g}^G$ appear and the sum is over $g \neq e$. This allows us to conclude that $\langle \psi_0 | \hat{\mathcal{S}}_X^{(i,k)} | \psi_0 \rangle = 0$, so that $\hat{\mathcal{S}}_X^{(i,k)}$ can also be used to detect the SPT order in $|\mathcal{C}\rangle$.

C. Topological Response

Topologically ordered and SPT states are known to pump quantized symmetry charge in response to the insertion of symmetry flux. This is known as *charge-flux attachment* [127] or *topological response* [128]. The most well known example of topological response is the quantum Hall effect, in which $U(1)$ symmetry flux in the form of magnetic flux induces an accumulation of $U(1)$ symmetry charge in the form of electric charge [126]. The quantized response in that case is the Hall conductivity. Topological response is an important signature which can distinguish SPT order from trivial order, the latter hosting no quantized response.

In the case of Abelian G , the response can be understood from the perspective of the decorated domain wall construction of the cluster state [147]. The wave function of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ cluster state can be understood as charges of the first \mathbb{Z}_2 decorated on the domain walls of the second \mathbb{Z}_2 . Threading a \mathbb{Z}_2 flux creates a single domain wall from which we can detect the charge of the other \mathbb{Z}_2 (see Appendix A 6 for further details). We will now see how this generalizes in the case of the G cluster state.

In order to define flux insertion for a 1d SPT, we place the state on a ring with periodic boundary conditions. Threading flux through a state on a ring is equivalent to twisting the boundary condition [129]. Threading a g -flux through $|\mathcal{C}\rangle$ has the effect of inserting a g domain wall between the first and last sites:

$$|\mathcal{C}_g\rangle := \mathcal{N} \sum_{g_i \in G} |g_1\rangle |g_1 \bar{g}_2\rangle |g_2\rangle \cdots |g_{N-1} \bar{g}_1\rangle, \quad (53)$$

where $|\mathcal{C}_g\rangle$ is the G cluster state on a ring with g flux inserted. Diagrammatically, this corresponds to inserting an X -type operator on the virtual space:

$$|\mathcal{C}_g\rangle = \boxed{\overleftarrow{X}_g} \bullet \circ \bullet \cdots \bullet \circ. \quad (54)$$

We now want to show that the system responds to the insertion of G flux by pumping a nontrivial $\text{Rep}(G)$ charge. The charge of $|\mathcal{C}\rangle$ under \hat{B}_Γ is 1 because $|\mathcal{C}\rangle$ is $\text{Rep}(G)$ -symmetric. If we act with \hat{B}_Γ on $|\mathcal{C}_g\rangle$ and find an eigenvalue other than 1, then we say that $|\mathcal{C}_g\rangle$ is nontrivially charged under the $\text{Rep}(G)$ symmetry. Indeed, we find

$$\hat{B}_\Gamma |\mathcal{C}_g\rangle = \text{Tr} \left[\prod_{i \text{ even}} Z_\Gamma^{(i)} \right] |\mathcal{C}_g\rangle = \text{Tr}[\Gamma(g)] |\mathcal{C}_g\rangle. \quad (55)$$

We see that the insertion of g -flux induces a response given by the character of g : $\text{Tr}[\Gamma(g)]$. This also implies the response only depends on the conjugacy class of g , since characters are invariant on conjugacy classes. It is also interesting to note that $|\mathcal{C}_g\rangle$ is an eigenstate of \hat{B}_Γ , i.e. it transforms as a 1-dimensional irrep of $\text{Rep}(G)$. In general, states which are charged under a symmetry G will transform as some irrep of G . In this case, the irreps of $\text{Rep}(G)$ are group elements due to Pontryagin duality: $\text{Rep}(\text{Rep}(G)) \cong G$. Each of these irreps is 1-dimensional, so $|\mathcal{C}_g\rangle$ must be an eigenstate of \hat{B}_Γ .

We can similarly thread a $\Gamma_{\alpha\beta}$ -flux through $|\mathcal{C}\rangle$ between the first and last sites, yielding

$$|\mathcal{C}_{\Gamma_{\alpha\beta}}\rangle := \mathcal{N} \sum_{g_i \in G} [\Gamma(g_1)]_{\alpha\beta} |g_1\rangle |g_1 \bar{g}_2\rangle |g_2\rangle \cdots |g_{N-1} \bar{g}_1\rangle. \quad (56)$$

Diagrammatically, this corresponds to inserting a Z -type operator in the virtual space:

$$|\mathcal{C}_\Gamma\rangle = \boxed{Z_\Gamma} \bullet \circ \bullet \cdots \bullet \circ. \quad (57)$$

We would like to understand how the state responds to the insertion of $\Gamma_{\alpha\beta}$ flux, so we act on it with \overleftarrow{A}_g , the representation of the G symmetry. We see that $|\mathcal{C}_{\Gamma_{\alpha\beta}}\rangle$ is transformed under this family of global X -type symmetries:

$$\begin{aligned} \overleftarrow{A}_g |\mathcal{C}_{\Gamma_{\alpha\beta}}\rangle &= \mathcal{N} \sum_{g_i \in G} [\Gamma(g_1 g)]_{\alpha\beta} |g_1\rangle |g_1 \bar{g}_2\rangle |g_2\rangle \cdots |g_{N-1} \bar{g}_1\rangle \\ &= \sum_{\gamma} [\Gamma(g)]_{\gamma\beta} |\mathcal{C}_{\Gamma_{\alpha\gamma}}\rangle \end{aligned} \quad (58)$$

Thus, $|\mathcal{C}_{\Gamma_{\alpha\beta}}\rangle$ is not necessarily an eigenstate of the G symmetry. Instead, we have a subspace supporting states

$$\left\{ \overleftarrow{A}_g |\mathcal{C}_{\Gamma_{\alpha\beta}}\rangle : g \in G \right\}. \quad (59)$$

This is sufficient to conclude that there is a nontrivial topological G response to the insertion of $\text{Rep}(G)$ flux.

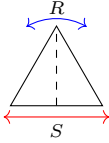
The product state $|\psi_0\rangle$ is unaffected by the insertion of symmetry flux. This can be understood as arising from the fact that $|0\rangle$ is a product state, with only single-site terms in its stabilizer Hamiltonian. Because there aren't any terms which span two sites, there are no terms to modify through flux insertion between the sites. Clearly, then, the product state does not have any charge response to the insertion of symmetry flux. This is yet another signature which distinguishes the state $|\mathcal{C}\rangle$ from the product state.

It was argued in [56] that charge and flux excitations in the quantum double bulk also yield twisted boundary conditions in the effective edge theory. Thus we can equivalently think of the probe in the above topological response as flux through the SPT on a ring, or as anyonic excitations in the bilayer quantum double which hosts $|\mathcal{C}\rangle$ as a gapped edge.

VI. DIHEDRAL EXAMPLE: $G = D_3$

We now turn to an example to illustrate the points of the previous section in a more concrete setting. We choose $G = D_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, as this is the smallest non-abelian group. It can be thought of as the symmetry group of an equilateral triangle, which consists of rotations by multiples of $2\pi/3$ ($R \in \mathbb{Z}_3$) and reflections ($S \in \mathbb{Z}_2$). It is a semidirect product, rather than a direct product, because the rotations and reflections do not commute. The group multiplication table is given by:

D_3	e	R	R^2	S	SR	SR^2
e	e	R	R^2	S	SR	SR^2
R	R	R^2	e	SR^2	S	SR
R^2	R^2	e	R	SR	SR^2	S
S	S	SR	SR^2	e	R	R^2
SR	SR	SR^2	S	R^2	e	R
SR^2	SR^2	S	SR	R	R^2	e



There are three irreducible representations of D_3 : the trivial irrep $\mathbf{1}$, the sign irrep Γ_s , and the 2d irrep Γ_{2d} . They are given by:

	$\mathbf{1}$	Γ_s	Γ_{2d}
e	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
R	1	1	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
R^2	1	1	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
S	1	-1	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
SR	1	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
SR^2	1	-1	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

We can also consider the tensor product of irreps, which decomposes into direct sums of irreps according to the rules of $\text{Rep}(D_3)$:

$$\text{Rep}(D_3) : \begin{array}{c|ccc} \otimes & \mathbf{1} & \Gamma_s & \Gamma_{2d} \\ \hline \mathbf{1} & \mathbf{1} & \Gamma_s & \Gamma_{2d} \\ \Gamma_s & \Gamma_s & \mathbf{1} & \Gamma_{2d} \\ \Gamma_{2d} & \Gamma_{2d} & \Gamma_{2d} & \mathbf{1} \oplus \Gamma_s \oplus \Gamma_{2d} \end{array}$$

Notice that $\mathbf{1}$ and Γ_s have inverses in the conventional sense, while Γ_{2d} does not. This means that $Z_{\Gamma_{2d}}$ does not have an inverse, so that the $\text{Rep}(D_3)$ symmetry is non-invertible. Namely,

$$\text{Tr}[Z_{\Gamma_{2d}}]\text{Tr}[Z_{\Gamma_{2d}}] = \text{Tr}[Z_{\mathbf{1}}] + \text{Tr}[Z_{\Gamma_s}] + \text{Tr}[Z_{\Gamma_{2d}}] \neq \mathbf{1}. \quad (60)$$

To see this explicitly, consider the $\text{Tr}[Z_{\Gamma_{2d}}]$ operator acting on a single site:

$$\text{Tr}[Z_{\Gamma_{2d}}] = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (61)$$

Clearly this matrix has determinant zero and is therefore not invertible.

A. Ground State Degeneracy

With this concrete model in hand, we explored several of the features discussed above numerically by constructing the Hamiltonian Eq. (27) and finding ground states using DMRG in ITensor [148]. We found $|D_3|^2 = 36$ ground states, consistent with the presence of a group-valued qudit at each edge.

We then tested the stability of the phase under random symmetric single-site perturbations. We formed the odd sublattice perturbations by first selecting a random Hermitian matrix \mathcal{O}_{odd} then symmetrizing it under D_3 :

$$\mathcal{O}_{\text{odd}}^{\text{sym}} = \sum_{g \in D_3} \hat{X}_g \mathcal{O}_{\text{odd}} \hat{X}_g^\dagger. \quad (62)$$

Symmetrizing an operator with respect to an MPO symmetry is more subtle, but one can derive using category theoretic methods that the most general single-site operator preserving $\text{Rep}(G)$ symmetry is a random real diagonal matrix [149]. We selected such a matrix $\mathcal{O}_{\text{even}}^{\text{sym}}$ as our even sublattice perturbation and considered the Hamiltonian

$$H'_C = H_C - \alpha \sum_{i \text{ odd}} \mathcal{O}_{\text{odd}}^{\text{sym}(i)} - \alpha \sum_{i \text{ even}} \mathcal{O}_{\text{even}}^{\text{sym}(i)}, \quad (63)$$

where α is the strength of the perturbation. For a symmetric perturbation with fixed small α , we expect that the ground state degeneracy is split by $\Delta \sim e^{-N}$ for finite system size N . In the thermodynamic limit $\Delta \rightarrow 0$ and the degeneracy is restored. This is illustrated in Fig. 3.

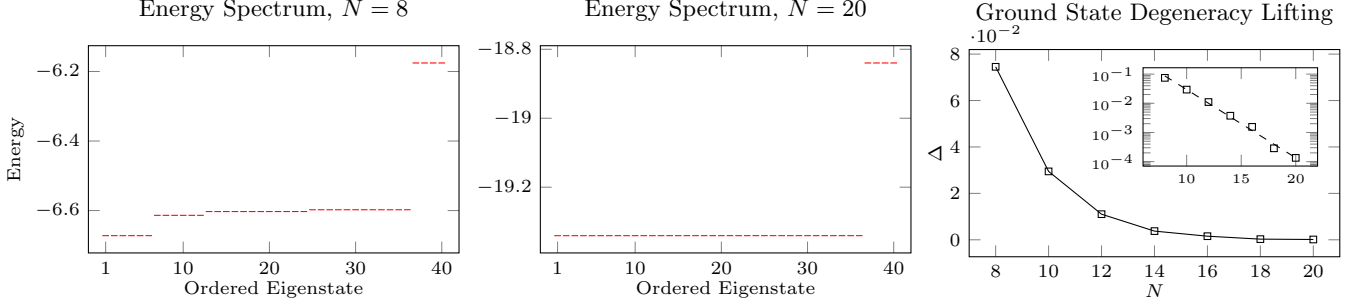


FIG. 3. **Perturbed H'_C .** We have numerically calculated the spectrum of H'_C on open chains of even sizes between $N = 8$ and $N = 20$, with the spectrum for the smallest and largest chains plotted. We have set $\alpha = 4 \cdot 10^{-2}$. The inset of the rightmost plot shows the same data on a log-linear scale with line of best fit in order to more obviously illustrate the exponential decay.

B. Entanglement Spectrum Degeneracy

We have also studied numerically the entanglement spectrum. Entanglement spectrum degeneracy is another signature of SPT order which arises from the projective action of the symmetry at the edge [150, 151]. For a bipartitioned state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the entanglement spectrum is defined as the eigenspectrum of the reduced density matrix $\rho_A = \text{Tr}_B[|\psi_{AB}\rangle\langle\psi_{AB}|]$.

We can identify the subsystems A and B with the left and right halves of an infinite chain. In order to calculate the entanglement spectrum of the chain, we would trace out one half of the chain, introducing a boundary of the remaining half-infinite chain where the two halves met. Because SPT states support unpinned degrees of freedom at their boundaries, we expect degeneracy in the energy spectrum of the remaining half-infinite chain, and therefore in the entanglement spectrum.

Numerically, we of course only have access to chains of finite length. However, we can think of the tensors deep in the bulk of a long-but-finite chain as approximating the tensors of an infinite chain, with the approximation converging as a function of system size. Accordingly, we can use a tensor deep in the bulk to calculate the entanglement spectrum degeneracy of the finite chain, the results of which are plotted in Fig. 4. In particular, we plot the normalized splitting between the largest eigenvalue λ_1 and sixth largest eigenvalue λ_6 in the entanglement spectrum of the ground state of the perturbed H'_C on open chains of various lengths. We find that this splitting tends to zero in the thermodynamic limit, indicative of a sixfold entanglement spectrum degeneracy. This is consistent with the physical picture of an unpinned D_3 -qudit hosted at the edge.

VII. REPEATED ACTION OF U_C

By repeatedly applying the state preparation unitary to $|\psi_0\rangle$, we can define a family of SPT states $|\mathcal{C}^n\rangle := (U_C)^n |\psi_0\rangle$ with the same symmetry and different projec-

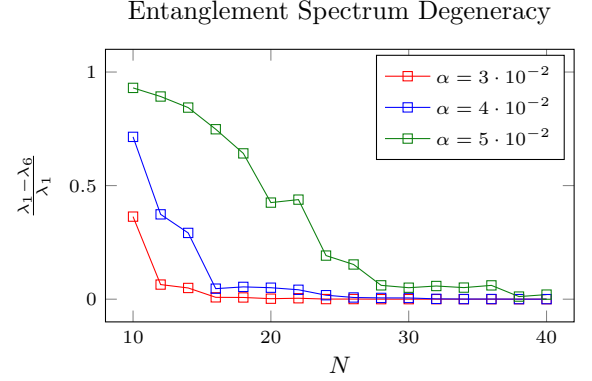


FIG. 4. Numerical results for the normalized splitting between the largest eigenvalue λ_1 and sixth largest eigenvalue λ_6 in the entanglement spectrum of the ground state of Eq. (63) on open chains of various lengths N , and as a function of the strength of symmetric noise α .

tive actions. We find

$$|\mathcal{C}^n\rangle = \mathcal{N} \sum_{\{g_i\}} |g_1\rangle |(g_1)^n (\bar{g}_2)^n| |g_2\rangle \cdots |(g_{N-1})^n (\bar{g}_N)^n| |g_N\rangle. \quad (64)$$

The MPS tensors for this state are given by

$$\text{---}\overset{\circ}{\bullet}\text{---} = \sum_g |g^n\rangle\langle g^n| \otimes |g\rangle, \quad \text{---}\overset{\circ}{\circ}\text{---} = \sum_g \vec{X}_g \otimes |g\rangle, \quad (65)$$

on odd and even sites respectively. The even sites are the same as those of $|\mathcal{C}\rangle$ and are denoted the same. The odd sites differ, so we introduce the above numbered tensor. This tensor satisfies

$$\begin{aligned} \text{---}\overset{\circ}{\bullet}\text{---} \otimes \vec{X}_g &= \text{---}\overset{\circ}{\bullet}\text{---} \otimes \vec{X}_g, \\ \text{---}\overset{\circ}{\circ}\text{---} \otimes \vec{X}_g &= \text{---}\overset{\circ}{\circ}\text{---} \otimes \vec{X}_g, \end{aligned}$$

This state is still $G_R \times \text{Rep}(G)$ -symmetric, with symmetry operators \hat{A}_g and \hat{B}_Γ . We can repeat the analysis of Sec. V A to determine the edge modes of this new state:

From this diagrammatic derivation, we can read off the edge modes

$$G : \hat{X}_{g^n}^{(L)} \hat{X}_{g^n}^{(R)}, \quad \text{Rep}(G) : [Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{\alpha\beta}, \quad (66)$$

which satisfy the commutation relations

$$\begin{aligned} \hat{X}_{g^n}^{(L)} [Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{\alpha\beta} &= [Z_\Gamma^{(L)} \cdot \Gamma(g^n) \cdot Z_\Gamma^{\dagger(R)}]_{\alpha\beta} \hat{X}_{g^n}^{(L)}, \\ \hat{X}_{g^n}^{(R)} [Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{\alpha\beta} &= [Z_\Gamma^{(L)} \cdot \Gamma(\bar{g}^n) \cdot Z_\Gamma^{\dagger(R)}]_{\alpha\beta} \hat{X}_{g^n}^{(R)}. \end{aligned} \quad (67)$$

Let n_* be the exponent of the group, defined to be the least common multiple of the order of each element in the group. This implies that $g^{n_*} = e$ for all $g \in G$, and n_* is the smallest integer for which this is true. When $n = n_*$, we have

$$|\mathcal{C}^{n_*}\rangle = |\mathbf{1}, e, \mathbf{1}, \dots, \mathbf{1}, e, \mathbf{1}\rangle = |\psi_0\rangle. \quad (68)$$

This shows that applying U_C to $|\psi_0\rangle$ n_* times returns $|\psi_0\rangle$. This can also be seen in the fact that the edge mode algebra Eq. (67) trivializes – that is, the edge operators all commute – when $n = n_*$. We can therefore create a family of n_* SPT states with symmetry $G_R \times \text{Rep}(G)$ by repeatedly applying U_C . In the case of Abelian G , these states give rise to distinct SPT phases. We leave the question of if and when these SPT states are all distinct to future work.

VIII. UTILITY FOR MEASUREMENT-BASED QUANTUM COMPUTATION

Measurement-based quantum computation (MBQC) is an alternative paradigm for quantum computing which

enacts unitary gates through measurement. Unlike traditional methods of quantum computation, MBQC begins with a highly entangled resource state, rather than generating entanglement during the computation. The choice of measurement basis enables the quantum programmer to implement particular unitary gates (often up to probabilistic byproducts) and is in general conditioned on outcomes of earlier measurements. Recently, it has also been shown that MBQC can be formulated in terms of a gauge theory [152].

Cluster states were the first states used for MBQC, and they remain the paradigmatic resource state [54, 132–134]. Later works have shown that the computational power of cluster states is intimately related to their SPT order, such that entire SPT phases are equivalent resource states for MBQC [93, 135, 137, 153, 154]. This observation has given rise to the notion of *computational phases of matter*, significantly expanding the conception of what can be an MBQC resource state. Because of the correspondence between SPT order and MBQC universality, we expect that the G cluster state should be useful as an MBQC resource state.

In this section, we will present an algorithm for MBQC on G cluster states based on abelian groups and non-abelian groups which are semidirect products of abelian groups, as well as lay out the idea to perform MBQC when G is a *solvable group*: a group formed via extensions of abelian groups. The abelian case is relatively straightforward, and we will show that the more complex cases can be understood as several parallel instances of the abelian case.

A. MBQC with MPS

The language of matrix product states naturally provides an elegant implementation of MBQC in 1d [130, 131]. This is called the correlation or virtual space picture of MBQC. In this picture, the logical qudit is the state in the virtual space encoding the boundary condition on the right end of the chain, and the computation takes place in the virtual space.

Consider an MPS

$$|\psi\rangle = \sum_{\{g_i\}} \langle v_L | A_{g_1}^{(1)} \cdots A_{g_n}^{(n)} | v_R \rangle |g_1, \dots, g_n\rangle, \quad (69)$$

where $|v_R\rangle$ is the right boundary condition and logical qudit. If we measure the physical qudit at site n and obtain the outcome $|s\rangle$, we find that the new state is

$$|\psi\rangle \xrightarrow{|s\rangle} \sum_{\{g_i\}} \langle v_L | A_{g_1}^{(1)} \cdots A_{g_{n-1}}^{(n-1)} A_s^{(n)} | v_R \rangle |g_1, \dots, g_{n-1}\rangle |s\rangle, \quad (70)$$

where $A_s^n = \sum_{g_n} \langle s | g_n \rangle A_{g_n}^{(n)}$. We can think of the $n-1$ qudits which we have not yet measured as an MPS

$$|\psi'\rangle = \sum_{\{g_i\}} \langle v_L | A_{g_1}^{(1)} \cdots A_{g_{n-1}}^{(n-1)} | v_R' \rangle |g_1, \dots, g_{n-1}\rangle, \quad (71)$$

where $|v'_R\rangle := A_s^{(n)}|v_R\rangle$.

We see that this measurement effectively acts on the virtual state $|v_R\rangle$ with the operator $A_s^{(n)}$, yielding the state $|v'_R\rangle := A_s^{(n)}|v_R\rangle$. The evolution from $|v_R\rangle$ to $|v'_R\rangle$ is one step of computation within the MBQC paradigm, akin to one layer of a quantum circuit. An MPS is said to be *universal* if this evolution can be used to apply any $SU(\chi)$ gate, where χ is the dimension of the virtual space.

B. G cluster state as an MBQC Resource State

Recall the explicit form of the MPS tensors for the G cluster state:

$$\text{---} \bullet \text{---} = \sum_{g \in G} |g\rangle \langle g| \otimes |g\rangle, \quad \text{---} \circ \text{---} = \sum_{g \in G} \vec{X}_g \otimes |g\rangle. \quad (72)$$

If we measure this state in the irrep matrix element basis on odd sites and group basis on even sites, we find the action on the virtual Hilbert space to be

$$\begin{aligned} \text{Odd: } |v_R\rangle &\xrightarrow{|\Gamma_{\alpha\beta}\rangle} [Z_\Gamma]_{\alpha\beta} |v_R\rangle, \\ \text{Even: } |v_R\rangle &\xrightarrow{|g\rangle} \vec{X}_g |v_R\rangle, \end{aligned} \quad (73)$$

where $|\Gamma_{\alpha\beta}\rangle$ and $|g\rangle$ are the measurement outcomes. Diagrammatically, we can write this as

$$\begin{aligned} \text{---} \circ \text{---} \xrightarrow{g} \text{---} \blacksquare &= \text{---} \blacksquare \xrightarrow{\vec{X}_g} \text{---} \blacksquare, & \text{---} \bullet \text{---} \xrightarrow{\Gamma_{\alpha\beta}} \text{---} \blacksquare &= \text{---} \blacksquare \xrightarrow{Z_\Gamma} \text{---} \blacksquare, \end{aligned} \quad (74)$$

This poses a complication: the Z -type Pauli errors are not in general unitary, and the X -type Pauli errors do not in general commute. This renders the straightforward generalization of the abelian protocol intractable. However, we find that the G cluster state is still universal for MBQC for a wide class of non-abelian groups, though some additional modifications from the abelian case must be made.

1. Warm-Up: G Abelian

We first show that the G cluster state is universal when the group is abelian. Here, we generalize the construction in Ref. 155 for performing MBQC with the qudit cluster state. We translate the protocol to the correlation space picture of MBQC and generalized from qudits ($G = \mathbb{Z}_d$) to arbitrary abelian G .

Implementing Gates Virtually. The universal single group-valued qudit gate set which we want to implement virtually is

$$R_G = \left\{ R_g^{(\pm)}(\theta), R_\Gamma^{(\pm)}(\theta) : g \in G, \Gamma \in \text{Rep}(G) \right\}, \quad (75)$$

where $\theta \in U(1)$ is a rotation angle. The rotation operators are given by

$$\begin{aligned} \vec{R}_g^{(\pm)}(\theta) &= \begin{cases} e^{i\theta(\vec{X}_g + \vec{X}_{\bar{g}})}, & g \neq \bar{g}, + \\ e^{-i\theta(\vec{X}_g - \vec{X}_{\bar{g}})}, & g \neq \bar{g}, - \\ e^{i\theta\vec{X}_g}, & g = \bar{g} \end{cases} \\ R_\Gamma^{(\pm)}(\theta) &= \begin{cases} e^{i\theta(Z_\Gamma + Z_\Gamma^\dagger)}, & \Gamma \text{ complex}, + \\ e^{-i\theta(Z_\Gamma - Z_\Gamma^\dagger)}, & \Gamma \text{ complex}, - \\ e^{i\theta Z_\Gamma}, & \Gamma \text{ real} \end{cases} \end{aligned} \quad (76)$$

We prove in Appendix H2 that this gate set is indeed universal for a single group-valued qudit.

We will now explain how to implement any gate in R_G on the virtual level by measuring the G cluster state in appropriately rotated bases. Essential to this protocol is the fact that we can implement these gates on the virtual level by applying an appropriate unitary to the physical legs. In particular, we find that

$$\begin{aligned} \text{---} \circ \text{---} \xrightarrow{\vec{R}_g^{(\pm)}} \text{---} \blacksquare &= \text{---} \circ \text{---} \xrightarrow{\vec{R}_g^{(\pm)}} \text{---} \blacksquare, \\ \text{---} \bullet \text{---} \xrightarrow{R_\Gamma^{(\pm)}} \text{---} \blacksquare &= \text{---} \bullet \text{---} \xrightarrow{R_\Gamma^{(\pm)}} \text{---} \blacksquare, \end{aligned} \quad (77)$$

where $\overleftarrow{R}_g^{(\pm)}$ is defined as in Eq. (76) but with right multiplication instead of left. These relations follow straightforwardly from Eq. (26a).

To implement the gate $\vec{R}_g^{(\pm)}(\theta)$ on the virtual level, we can measure an even site in the basis $\{\overleftarrow{R}_g^{(\pm)\dagger}(\theta) |h\rangle : h \in G\}$:

$$\text{---} \circ \text{---} \xrightarrow{\overleftarrow{R}_g^{(\pm)}} \text{---} \blacksquare = \text{---} \circ \text{---} \xrightarrow{h} \text{---} \blacksquare \xrightarrow{\vec{R}_g^{(\pm)}} \text{---} \blacksquare \quad (78)$$

Similarly, to implement the gate $R_\Gamma^{(\pm)}(\theta)$ on the virtual level, we can measure an odd site in the basis⁴ $\{R_\Gamma^{(\pm)\dagger}(\theta) |\Gamma_x\rangle : x \in G\}$:

$$\text{---} \bullet \text{---} \xrightarrow{R_\Gamma^{(\pm)}} \text{---} \blacksquare = \text{---} \bullet \text{---} \xrightarrow{\Gamma_x} \text{---} \blacksquare \xrightarrow{R_\Gamma^{(\pm)}} \text{---} \blacksquare \quad (79)$$

⁴ Because the group is abelian, the set of irreducible representations form a group isomorphic to the original group and can thus be labeled by the group elements. This is the reasoning for the irrep notation Γ_x : Γ_x is the irrep that is associated with $x \in G$ through the Fourier transform.

Recalling that measuring the even and odd MPS tensors has the following action:

$$\begin{array}{c} \overline{g} \\ \hline \bigcirc \end{array} = \boxed{\overrightarrow{X}_g}, \quad \begin{array}{c} \overline{\Gamma} \\ \hline \bullet \end{array} = \boxed{Z_\Gamma}, \quad (80)$$

we see that the total effect of the rotated measurement on the boundary state is

$$|v_R\rangle \xrightarrow{\overleftarrow{R}_g^{(\pm)\dagger} \otimes R_\Gamma^{(\pm)\dagger} |h, \Gamma_x\rangle} \overrightarrow{X}_h \overrightarrow{R}_g^{(\pm)} Z_{\Gamma_x} R_\Gamma^{(\pm)} |v_R\rangle. \quad (81)$$

We are therefore able to deterministically implement the entire gate set R_G up to group-based Pauli byproducts which are dependent on the measurement outcomes.

Adaptive Measurements. Notice that Pauli byproducts lie between the logical gates in Eq. (81). We would prefer for all logical gates to be to the right, acting on the logical qudit, with Pauli by-products to the left. We can ensure this form by adapting later measurement bases to compensate for earlier Pauli errors.

Consider an example with three sites which illustrates the general strategy. The first measurement is performed on an odd site in the usual basis:

$$\dots \bullet \circ \bullet \blacksquare \mapsto \dots \bullet \circ \bullet \begin{array}{c} \overline{\Gamma_x} \\ \hline \boxed{R_\Gamma^{(\pm)}} \end{array} \blacksquare = \dots \bullet \circ \bullet \begin{array}{c} \overline{\Gamma_x} \\ \hline \boxed{R_\Gamma^{(\pm)}} \end{array} \blacksquare = \dots \bullet \circ \boxed{Z_{\Gamma_x}} \boxed{R_\Gamma^{(\pm)}} \blacksquare. \quad (82)$$

In order to compensate for the Z_{Γ_x} error on the virtual level, we measure in the basis $\left\{ \left(\overleftarrow{R}_g^{(\pm)}(\theta) Z_{\Gamma_x} \right)^\dagger |h\rangle : h \in G \right\}$, which will propagate the error past the next logical operation:

$$\mapsto \dots \bullet \circ \begin{array}{c} \overline{h} \\ \hline \boxed{\overleftarrow{R}_g^{(\pm)}} \\ \hline \boxed{Z_{\Gamma_x}} \end{array} \boxed{R_\Gamma^{(\pm)}} \blacksquare = \dots \bullet \circ \begin{array}{c} \overline{h} \\ \hline \boxed{\overleftarrow{R}_g^{(\pm)}} \end{array} \boxed{R_\Gamma^{(\pm)}} \blacksquare = \dots \bullet \circ \boxed{Z_{\Gamma_x}} \boxed{\overrightarrow{X}_h} \boxed{\overrightarrow{R}_g^{(\pm)}} \boxed{R_\Gamma^{(\pm)}} \blacksquare$$

We can use the properties of the tensors to move the Z_{Γ_x} to the left of the next odd tensor, after which we can make a measurement in the basis $\left\{ \left(R_{\Gamma'}^{(\pm)}(\theta) \overrightarrow{X}_h \right)^\dagger |h\rangle : h \in G \right\}$ which moves the \overrightarrow{X}_h error to the left:

$$\begin{aligned} &= \dots \circ \boxed{Z_{\Gamma_x}} \bullet \boxed{\overrightarrow{X}_h} \boxed{\overrightarrow{R}_g^{(\pm)}} \boxed{R_\Gamma^{(\pm)}} \blacksquare \mapsto \dots \circ \boxed{Z_{\Gamma_x}} \bullet \begin{array}{c} \overline{\Gamma_y} \\ \hline \boxed{R_{\Gamma'}^{(\pm)}} \\ \hline \boxed{\overrightarrow{X}_h} \end{array} \boxed{\overrightarrow{R}_g^{(\pm)}} \boxed{R_\Gamma^{(\pm)}} \blacksquare \\ &= \dots \circ \begin{array}{c} \overline{\Gamma_y} \\ \hline \boxed{R_{\Gamma'}^{(\pm)}} \end{array} \bullet \boxed{\overrightarrow{X}_h} \boxed{\overrightarrow{R}_g^{(\pm)}} \boxed{R_\Gamma^{(\pm)}} \blacksquare = \Gamma_x(h) \dots \circ \begin{array}{c} \overline{\Gamma_y} \\ \hline \boxed{R_{\Gamma'}^{(\pm)}} \end{array} \bullet \boxed{\overrightarrow{X}_h} \boxed{Z_{\Gamma_{xy}}} \boxed{R_{\Gamma'}^{(\pm)}} \boxed{\overrightarrow{R}_g^{(\pm)}} \boxed{R_\Gamma^{(\pm)}} \blacksquare, \end{aligned}$$

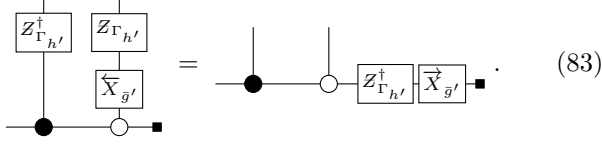
where we have used the group-based Pauli commutation relation $Z_{\Gamma_x} \overrightarrow{X}_g = \Gamma_x(g) \overrightarrow{X}_g Z_{\Gamma_x}$ and multiplication of abelian irreps $Z_{\Gamma_x} Z_{\Gamma_y} = Z_{\Gamma_{xy}}$. Because we have taken G to be abelian, the factor $\Gamma_x(h)$ is a global phase which can be discarded.

We have demonstrated the essential aspects of the strategy, which can be employed indefinitely to perform any number of logical operations. As desired, we see

that conditioning measurement bases on the outcomes of previous measurements allows us to propagate all Pauli errors to the left of the logical operators.

Outcome of Computation. Once we have applied the entire sequence of gates, we will have mapped the edge state from $|v_R\rangle$ to $|v_{R,\text{target}}\rangle$, up to a global phase which we can disregard and a Pauli byproduct $\overrightarrow{X}_{g'} Z_{\Gamma_{h'}}$. g' and h' will be a function of the measurement outcomes

and entirely determined by the relations Eq. (26a). To correct this Pauli error and complete the computation, we can use classical post-processing, or if available experimentally, act on the physical legs of the next measurement site with $(Z_{\Gamma_{h'}}^{(\bullet)\dagger} \cdot Z_{\Gamma_{h'}}^{(\circ)}) (1 \otimes \vec{X}_{\vec{g}'})$, which pulls through to $Z_{\Gamma_{h'}}^\dagger \vec{X}_{\vec{g}'}$:



In summary, we have shown that the abelian G cluster state is a suitable resource state for universal single group-valued qudit MBQC. In particular, we can implement each gate in the universal gate set R_G (up to group-based Pauli byproducts) through measurements in rotated bases. We have shown that we can use knowledge of measurement results and classical feedback to propagate byproducts to the left and apply a series of gates sequentially. In the next section, we will show that only small modifications are necessary to adapt this protocol to utilize certain non-abelian G cluster states for MBQC.

2. G a Semidirect Product of Abelian Groups

We approach non-abelian groups by considering those which can be constructed as semidirect products of abelian groups. Recently, a systematic construction of topological order preparation through finite-depth local unitaries and measurements was put forth in Ref. 86. En route to their results, the authors present a general formalism for implementing KW duality on a solvable group by applying the duality to a sequence of abelian subgroups. In this construction, unitaries encoding the particular extension are intertwined between abelian KW MPOs to form the non-abelian KW MPO. Groups constructed as semidirect products of abelian groups are a special case of solvable groups, so we may apply these results to the present task.

Recalling that the KW MPO is simply the cluster state MPS with odd physical legs flipped [145], this construction in turn provides a construction of the non-abelian G cluster state in terms of abelian cluster states and unitaries (see Eq. (86) for a special case). We will use this decomposition to build up MBQC with non-abelian G cluster states from the abelian computation protocol laid out in the preceding section.

A particularly simple class of non-abelian groups are those which can be constructed as a *split extension* of two abelian groups. Mathematically, we say that some group G is an *extension* of a quotient group Q by a normal subgroup N if there exists a short exact sequence

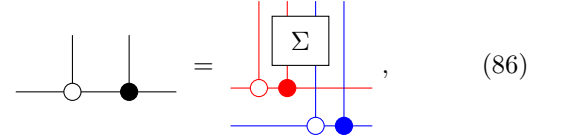
$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1. \quad (84)$$

We say that the extension is *split* if G is a semidirect product of Q and N . In this case, the decomposition of a non-abelian G -qudit into abelian N - and Q -qudits is relatively simple, as presented in Eq. (86).

As Hilbert spaces, there exists an isomorphism $\mathbb{C}[G] \cong \mathbb{C}[Q] \otimes \mathbb{C}[N]$ which enables the decomposition

$$|g\rangle \rightarrow |q_g, n_g\rangle. \quad (85)$$

However, this does not yet capture the fact that G is a *semidirect* product of Q and N . This is encoded in the action of Q on N given by a map $\sigma : Q \rightarrow \text{Aut}(N)$ (i.e., $\sigma^q : N \rightarrow N$ is an automorphism). We may define a unitary operator which implements this group action as a controlled operator $\Sigma = \sum_{q,n} |q\rangle\langle q| \otimes \sigma^q[n]$. One can show that using this unitary, the MPS tensors can be decomposed as [86]



where the red and blue MPSs are group cluster states corresponding to the abelian groups Q and N respectively:

$$\begin{aligned} \text{red leg} &= \sum_{q \in Q} |q\rangle\langle q| \otimes |q\rangle, & \text{red leg} &= \sum_{q \in Q} \vec{X}_q \otimes |q\rangle, \\ \text{blue leg} &= \sum_{n \in N} |n\rangle\langle n| \otimes |n\rangle, & \text{blue leg} &= \sum_{n \in N} \vec{X}_n \otimes |n\rangle. \end{aligned} \quad (87)$$

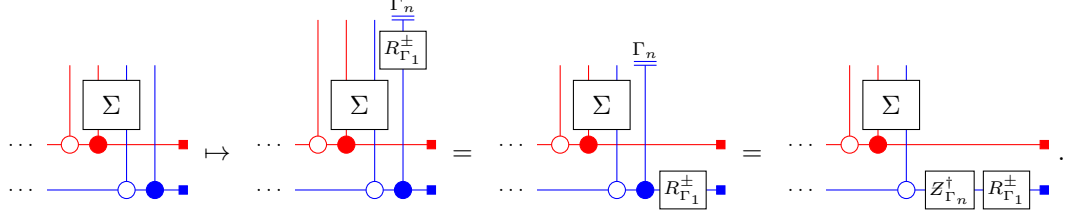
In order to compensate for the additional Σ operator, we will only have to slightly modify the approach we took in the abelian case above.

Due to the decomposition $\mathbb{C}[G] = \mathbb{C}[Q] \otimes \mathbb{C}[N]$ we may perform a measurement in each factor separately.

We first measure the blue odd site in the basis:

$$\left\{ (R_{\Gamma_1}^\pm)^\dagger |\Gamma_n\rangle : \Gamma_n \in \text{Rep}(N) \right\}. \quad (88)$$

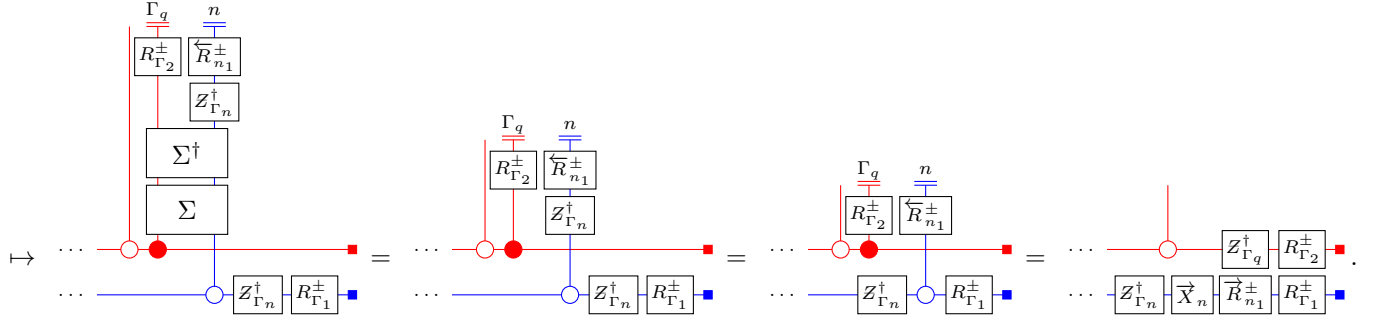
The effect of this measurement is



We next measure the two central qudits so that we can explicitly counteract the gate Σ . We measure in the basis

$$\left\{ \left[\left(R_{\Gamma_2}^\pm \otimes \overleftarrow{R}_{n_1}^\pm \right) \left(\mathbb{1} \otimes Z_{\Gamma_n}^\dagger \right) \Sigma^\dagger \right]^\dagger |\Gamma_q, n\rangle : \Gamma_q \in \text{Rep}(Q), n \in N \right\}. \quad (89)$$

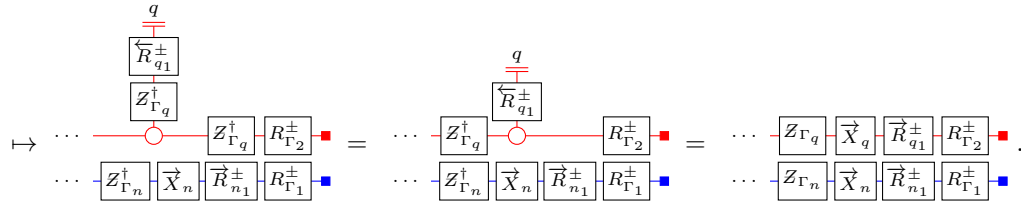
The effect of this measurement is



Finally, we measure the red even site in the basis:

$$\left\{ \left(\overleftarrow{R}_{q_1}^\pm Z_{\Gamma_q}^\dagger \right)^\dagger |q\rangle : q \in Q \right\}. \quad (90)$$

The effect of this measurement is



We now see that we can perform universal single-qudit quantum computation in the Q and N virtual spaces separately. Thus far, the only difference from the abelian case of Sec. VIII B 1 is the additional rotation by Σ^\dagger in the measurement basis on the center two sites, which was necessary to compensate for the Σ gate.

However, in order to achieve single-qudit universality on the G qudit, we must achieve two-qudit universality on the pair of Q and N qudits. Performing universal single-qudit rotations on the Q and N virtual spaces is not sufficient; we must also introduce a two-qudit imprimitive gate – a gate which generates entanglement – acting between the Q and N virtual spaces.

The imprimitive gate which we define is a controlled-

multiplication acting between the groups:

$$\begin{aligned} \overrightarrow{\Lambda} &:= \sum_q |q\rangle\langle q| \otimes \overrightarrow{X}_{\lambda(q)}, \\ \overleftarrow{\Lambda} &:= \sum_q |q\rangle\langle q| \otimes \overleftarrow{X}_{\lambda(q)}, \end{aligned} \quad (91)$$

where $\lambda : Q \rightarrow N$ is a map from Q to N which generates entanglement. Notice that $\overleftarrow{\Lambda}$ acting on the two central qudits can be pulled from the physical legs to the virtual:

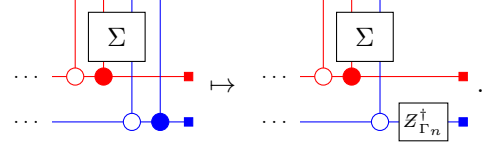
$$\overleftarrow{\Lambda} = \overrightarrow{\Lambda} \quad (92)$$

We can repeat the preceding analysis and show that we can enact $\vec{\Lambda}$ on the virtual space via an appropriate series of measurements.

We first measure the odd blue site in the basis

$$\{|\Gamma_n\rangle : \Gamma_n \in \text{Rep}(N)\}, \quad (93)$$

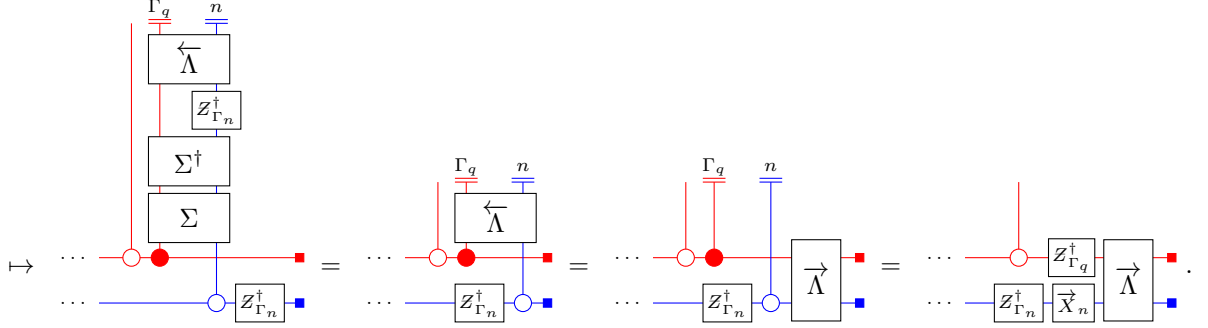
resulting in the following action:



Next, we measure the two center sites in the basis

$$\left\{ \left[\overleftarrow{\Lambda} \left(\mathbb{1} \otimes Z_{\Gamma_n}^\dagger \right) \Sigma^\dagger \right]^\dagger |\Gamma_q, n\rangle : \Gamma_q \in \text{Rep}(Q), n \in N \right\}, \quad (94)$$

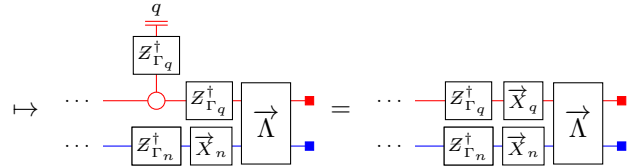
resulting in



Finally, we measure the remaining red even site in the basis:

$$\left\{ \left(Z_{\Gamma_q}^\dagger \right)^\dagger |q\rangle : q \in Q \right\}, \quad (95)$$

resulting in



We now see that we can implement $\vec{\Lambda}$ virtually by measuring the center two sites in the basis Eq. (94). We prove in Appendix H 1 that this gate is imprimitive so long as $\lambda(q)$ depends non-trivially on q .

In summary, we have shown that MBQC in the case where G is a split extension of abelian groups can be understood as two instances of the abelian MBQC protocol running in parallel. We saw that we can compensate for the entangling gate Σ in our choice of measurement bases and that we can generate entanglement between the two abelian MBQC rails via the gate $\vec{\Lambda}$. It is also possible to treat the case where G is a successive split extension involving several abelian groups similarly, which is depicted schematically in Fig. 5.

3. G Solvable

If G is solvable, it can be constructed from abelian groups using extensions. The example discussed in the previous section – namely, a group G formed via a split extension of the abelian groups Q and N – is a simple example of a solvable group. For any solvable group, the cluster state MPS tensors can be decomposed into a set of abelian MPS tensors (corresponding to the set of abelian groups) and unitaries (capturing the details of the extensions of these abelian groups), generalizing Eq. (86). In general, this decomposition will be of the form Fig. 6.

It is not obvious how to generalize the strategy discussed in the previous section to this class of groups be-

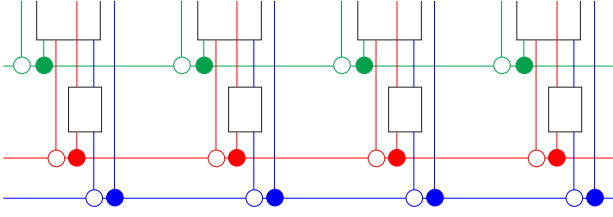


FIG. 5. MPS decomposition for the G cluster state based on a group constructed from split extensions of abelian groups. Different colored tensors are cluster states corresponding to different abelian groups and odd sites are control qudits for even site(s).

cause the unitaries in the decomposition have overlapping support. However, because cluster states derived from solvable groups still admit a decomposition into abelian groups and unitaries, we suspect that it should be possible to devise a protocol which performs universal MBQC using such a state as resource. We leave the details of this construction to future work.

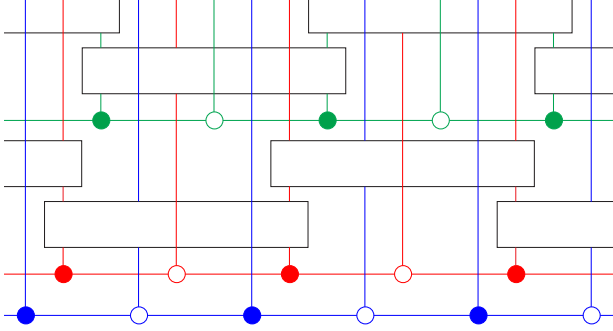


FIG. 6. MPS decomposition for the G cluster state based on some solvable group. Once again, different colored tensors are cluster states corresponding to different abelian groups and odd sites are control qudits for even sites.

4. G not Solvable

In the case where G is not solvable, we cannot decompose G into a set of abelian groups. We are therefore met with the same problems which arise when trying to design an MBQC algorithm for non-abelian G directly: the Z -type Pauli errors are not in general unitary, and the X -type Pauli errors do not in general commute. We leave the challenge of designing an MBQC algorithm for these cases to future work.

IX. DISCUSSION AND OUTLOOK

We have studied the G cluster state as an SPT state protected by a fusion category symmetry. Our contribu-

tion to the literature is a study of the microscopic signatures of fusion category SPT order in a relatively simple lattice model. This allows us to explore fusion category SPT order in greater detail and with more direct applicability to quantum information applications than in prior works. This study would be useful in developing a general microscopic theory of fusion category SPTs as successful as that of group SPTs. It also presents several technical contributions, including graphical methods for manipulating the group-based Pauli operators which could be useful in the study of other exotic lattice models.

Group-Based Pauli Spin Chains. Pauli spin chains are extremely versatile toy models which, despite their simplicity, exhibit many interesting phenomena. As we have pointed out, relatively few of these phenomena have been generalized to group-valued Hilbert spaces. It is beneficial to do so not only to study fusion category symmetry as we have done, but also to design systems with on-demand finite group symmetries, to study gauge theories, or to design more complex quantum computational schemes. Group-based Pauli spin chains also retain the properties that have made Pauli spin chains so ubiquitous: they are simple to write down and highly versatile.

The techniques we have used to study the G cluster state are highly applicable in the theoretical analysis of other group-based Pauli spin chains. We have here developed both diagrammatic and algebraic methods for working with group-based Pauli spin chains, and have already seen how to generalize symmetries, stabilizers, and edge modes.

With spin degrees of freedom and operators comprised of Pauli strings, spin chains are also natural models to study on quantum hardware. This is particularly relevant for researchers interested in quantum simulation and quantum computing. Group-based spin chains comprise a class of exotic models which could be realized in a quantum simulator. A proposal already exists for encoding qudits valued in the dihedral group D_3 into quantum hardware [156]. Such proposals also allow for any quantum computational protocols designed using group-valued qudits to be implemented directly.

Algebraic Classification of \mathcal{A} -SPTs. While classifications of fusion category SPTs in terms of categorical data exist [60, 62], it is not yet known whether fusion categories may be classified by an algebraic object (such as a cohomology group), as is the case for many conventional SPTs. The classification of 1d bosonic SPTs protected by group symmetries in terms of group cohomology [52, 114] has been extremely successful. It has been essential in formulating the connections between SPTs and measurement-based quantum computing [135, 137] and the generation of long-range entanglement [145], as well as in placing SPT states within the modern categorical understanding of topological order [47]. It is therefore a very interesting open question to determine whether SPTs protected by categorical symmetries can be classified in terms of some algebraic object. Our discussion of edge modes in Sec. V A and of constructing

other $G \times \text{Rep}(G)$ -SPTs in Sec. VII should shed light on this search, just as similar inquiries in group-symmetric SPTs point toward the group cohomology classification.

We suspect that it should be possible to derive a classification of fusion category SPTs in terms of some algebraic object using the methods developed in Ref. 157. This paper considers MPSs symmetric under the action of MPOs, which can be cast into the language of bimodule categories. By choosing appropriate categories to describe finite group symmetry as an MPO, this formalism recovers the classification in terms of group cohomology as the solution to a coupled pentagon equation. Repeating this procedure with a different choice of categories, chosen to describe fusion category symmetry as an MPO, will return a different constraint. We conjecture that this will be a generalized cocycle condition which should be satisfied by our matrix “phases” $\Gamma(g)$. With this condition in hand, it may then be possible to recognize it as defining some algebraic object, just as the 2-cocycle condition (and 2-coboundaries) define the second cohomology group. It would also be of significant practical value to understand how such an object is manifest microscopically, if one exists.

G cluster state as Gapped Boundary of 2d TO. We expect that the G cluster state $|\mathcal{C}\rangle$ can be derived as a gapped boundary of the quantum double $\mathcal{D}(G \times \text{Rep}(G))$. It is known that the qubit cluster state can be realized as the gapped boundary of the bilayer toric code where $e_1 m_2$ and $m_1 e_2$ anyons are condensed [5, 158]. Similarly, the CSS cluster state corresponds to the boundary where $e_1 e_2$ and $m_1 m_2$ are condensed. More generally for group G case, we know that H_{SSB2} in Eq. (34) corresponds to the gapped boundary of $\mathcal{D}(G \times G)$ where $a_1 \bar{a}_2$ condenses for all $a \in \mathcal{D}(G)$ (which corresponds to the trivial interface between $\mathcal{D}(G)$ and itself after unfolding [159]. Performing an isomorphism relating anyons in $\mathcal{D}(G)$ to $\mathcal{D}(\text{Rep}(G))$ on the second copy gives a construction of the G cluster state as a gapped boundary of $\mathcal{D}(G \times \text{Rep}(G))$. It would be interesting to work out the microscopics for this construction.

$|\mathcal{C}\rangle$ as an Error Correcting Code. We have repeatedly referred to $|\mathcal{C}\rangle$ as a generalized stabilizer state, and it would be interesting to study it as a generalized stabilizer error correcting code. One immediate complication is that the group-based Pauli operators \vec{X}_g , \vec{Y}_g , and Z_Γ no longer form a group when G is non-abelian. It is unclear what algebraic object these operators form under multiplication, and this question is also directly related to the classification of the edge modes, as the edge modes form the same object.

Despite these complications, the stabilizers still serve

to uniquely specify the G cluster state and could therefore protect information in a well-defined codespace. We leave to future work the task of rigorously defining group-based Pauli stabilizer codes and exploring where they may offer an advantage over traditional stabilizer codes.

$|\mathcal{C}\rangle$ as a Higgs phase of a gauge theory. It has been recently appreciated that the \mathbb{Z}_d cluster state is naturally thought of as the Higgs phase of a \mathbb{Z}_d gauge theory [160], where the even/odd sites corresponds to \mathbb{Z}_d matter/gauge fields, respectively. The two stabilizers of the cluster state can then be thought of as imposing a Gauss law of the gauge theory, while the other realizes the Higgs condensation by proliferating matter fields connected by Wilson loops. It would be interesting to establish this viewpoint for non-Abelian groups as well, both in terms of a G -gauge theory, and perhaps more interestingly in terms of a non-invertible gauge symmetry $\text{Rep}(G)$.

Group-based cluster states in higher dimensions as SPT phases. We expect the analysis in higher dimensions to be straightforward. In particular, in d spatial dimensions, one can construct a generalized cluster state by putting Z -type tensors on vertices and X -type tensors on directed edges of a cellulation of the manifold. We expect that this cluster state is an SPT state protected by $G \times \text{Rep}(G)^{(d-1)}$ symmetry, where $\text{Rep}(G)^{(d-1)}$ is a $(d-1)$ -form non-invertible symmetry.

$|\mathcal{C}\rangle$ for Hopf Algebras as SPT phases. In general, generalized cluster states can be defined for any Hopf algebra [55]. Unlike typical anyon chains [161], such models live in a local tensor product Hilbert space. It would be interesting to analyze the SPT properties of these states in full generality.

ACKNOWLEDGMENTS

We thank Dave Aasen, Maissam Barkeshli, Wenjie Ji, Laurens Lootens, David T. Stephen, Ruben Verresen, and Brayden Ware for helpful discussions. CF thanks the Joint Quantum Institute at the University of Maryland for support through a JQI fellowship. CF was supported in part by the NSF STAQ program. NT is supported by the Walter Burke Institute for Theoretical Physics at Caltech. VVA acknowledges support from NSF QLCI grant OMA-2120757. Contributions to this work by NIST, an agency of the US government, are not subject to US copyright. Any mention of commercial products does not indicate endorsement by NIST. VVA thanks Olga Albert and Ryhor Kandratsenia for providing daycare support throughout this work.

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Appendix A: Qubit CSS Cluster State

In this section, we briefly review the CSS cluster state defined on qubits and derive several results which are generalized in the main text.

1. State and MPS

In the case $G = \mathbb{Z}_2$, the G cluster state reduces to the qubit CSS cluster state:

$$|\text{CSS}\rangle = 2^{-N/2} \sum_{\{g_i\} \in \mathbb{Z}_2} |g_1\rangle |g_1 \oplus g_2\rangle |g_2\rangle \cdots |g_{N-1} \oplus g_N\rangle |g_N\rangle, \quad (\text{A1})$$

where \oplus denotes addition modulo 2. This state can be written as an MPS with tensors

$$\text{---} \bullet \text{---} = |0\rangle\langle 0| \otimes |0\rangle + |1\rangle\langle 1| \otimes |1\rangle, \quad \text{---} \circ \text{---} = \mathbb{1} \otimes |0\rangle + X \otimes |1\rangle. \quad (\text{A2})$$

These tensors satisfy the following identities:

$$\begin{aligned} \text{---} \bullet \text{---} \begin{array}{c} | \\ \boxed{X} \end{array} &= |0\rangle\langle 0| \otimes |1\rangle + |1\rangle\langle 1| \otimes |0\rangle = \text{---} \boxed{X} \bullet \boxed{X} \text{---}, \\ \text{---} \circ \text{---} \begin{array}{c} | \\ \boxed{X} \end{array} &= \mathbb{1} \otimes |1\rangle + X \otimes |0\rangle = \text{---} \boxed{X} \circ \text{---} = \text{---} \circ \boxed{X} \text{---}, \\ \text{---} \bullet \text{---} \begin{array}{c} | \\ \boxed{Z} \end{array} &= |0\rangle\langle 0| \otimes |0\rangle - |1\rangle\langle 1| \otimes |1\rangle = \text{---} \boxed{Z} \bullet \text{---} = \text{---} \bullet \boxed{Z} \text{---}, \\ \text{---} \circ \text{---} \begin{array}{c} | \\ \boxed{Z} \end{array} &= \mathbb{1} \otimes |0\rangle - X \otimes |1\rangle = \text{---} \boxed{Z} \circ \text{---} = \text{---} \circ \boxed{Z} \text{---}, \\ \text{---} \boxed{Z} \bullet \boxed{Z} \text{---} &= |0\rangle\langle 0| \otimes |0\rangle + |1\rangle\langle 1| \otimes |1\rangle = \text{---} \bullet \text{---}, \\ \text{---} \boxed{X} \circ \boxed{X} \text{---} &= \mathbb{1} \otimes |0\rangle + X \otimes |1\rangle = \text{---} \circ \text{---}. \end{aligned}$$

The corresponding product state is $|\psi_0\rangle = |+, 0, +, 0, \dots\rangle$.

2. Hamiltonian and Symmetries

The qubit CSS cluster state is the ground state of the commuting stabilizer Hamiltonian

$$H = - \sum_{i \text{ odd}} X^{(i+1)} X^{(i+2)} X^{(i+3)} + Z^{(i)} Z^{(i+1)} Z^{(i+2)}. \quad (\text{A3})$$

This Hamiltonian is obtained from the usual cluster state stabilizers by performing a Hadamard gate on the middle qubit of each string. This Hamiltonian has global symmetries

$$\hat{A} = \prod_{i \text{ odd}} X^{(i)}, \quad \hat{B} = \prod_{i \text{ even}} Z^{(i)}. \quad (\text{A4})$$

These symmetries commute with one another and each square to 1. Together, they form a representation of the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

From this diagrammatic derivation, we can read off the edge modes

$$\hat{A} : X^{(L)} X^{(R)}, \quad \hat{B} : Z^{(L)} Z^{(R)} \quad (\text{A10})$$

acting on the edge Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_R$.

Notice that

$$\{X^{(L)}, Z^{(L)}\} = \{X^{(R)}, Z^{(R)}\} = 0, \quad (\text{A11})$$

so that the left and right operators generate the single-qubit Pauli algebra on the left and right ends of the chain, respectively. Recognizing that the Pauli matrices are a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have found edge modes transforming as a projective representation of the protecting symmetry, as is expected in a 1d SPT phase with on-site group symmetry.

6. Topological Response

There is only a single nontrivial \mathbb{Z}_2 -flux state, which corresponds to $g = 1$:

$$|\text{CSS}_1\rangle := 2^{-(N-1)/2} \sum_{g_i \in \mathbb{Z}_2} |g_1\rangle |g_1 \oplus g_2\rangle |g_2\rangle \cdots |g_{N-1} \oplus (g_1 \oplus 1)\rangle. \quad (\text{A12})$$

This state is obtained from $|\text{CSS}\rangle$ by twisting the boundary condition from one end of the chain to the other. It acquires a nontrivial charge under \hat{B} :

$$\hat{B}|\text{CSS}_1\rangle = (-1)^{g_1+g_2+g_2+\cdots+g_{N-1}+g_{N-1}+g_1+1} |\text{CSS}_1\rangle = -|\text{CSS}_1\rangle, \quad (\text{A13})$$

where we have used the fact that $2g_i$ is even for all g_i in \mathbb{Z}_2 . Similarly, there is only a single nontrivial $\text{Rep}(\mathbb{Z}_2)$ -flux state corresponding to the sign irrep $\Gamma = \Gamma_-$:

$$|\text{CSS}_{\Gamma_-}\rangle = 2^{-(N-1)/2} \sum_{\{g_i\} \in \mathbb{Z}_2} \Gamma_-(g_1) |g_1\rangle |g_1 \oplus g_2\rangle |g_2\rangle \cdots |g_{N-1} \oplus g_1\rangle. \quad (\text{A14})$$

We find that this state is nontrivially charged under \hat{A} :

$$\begin{aligned} \hat{A}|\text{CSS}_{\Gamma_-}\rangle &= 2^{-(N-1)/2} \sum_{\{g_i\} \in \mathbb{Z}_2} \Gamma_-(g_1) |g_1 \oplus 1\rangle |g_1 \oplus g_2\rangle |g_2 \oplus 1\rangle \cdots |g_{N-1} \oplus g_1\rangle \\ &= 2^{-(N-1)/2} \sum_{\{g_i\} \in \mathbb{Z}_2} \Gamma_-(g_1 \oplus 1) |g_1\rangle |g_1 \oplus g_2\rangle |g_2\rangle \cdots |g_{N-1} \oplus g_1\rangle = -|\text{CSS}_{\Gamma_-}\rangle. \end{aligned} \quad (\text{A15})$$

Because \mathbb{Z}_2 is abelian, it is easy to see that the response is symmetric. That is, we find a \mathbb{Z}_2 charge of -1 in response to $\text{Rep}(\mathbb{Z}_2)$ flux, and a $\text{Rep}(\mathbb{Z}_2)$ charge of -1 in response to \mathbb{Z}_2 flux.

The response for the cluster state can be understood from the perspective of charges of one \mathbb{Z}_2 symmetry decorated on the domain wall of the other \mathbb{Z}_2 symmetry. Inserting a flux generates a single domain wall from which we can see the charge response [147].

Appendix B: Additional Symmetry

In this section, we seek to better understand the full symmetry group of the G cluster state. There are four symmetries acting independently, so that the total symmetry group is $G_L \times G_R \times \text{Rep}(G) \times \text{Inn}(G)$. We would like to understand which symmetries are actually necessary for protection of the SPT phase. For example, it is insightful to recall that the AKLT model [162] is a non-trivial SPT phase when protected separately either under $\text{SO}(3)$ rotations or time reversal, meaning that weak local perturbations which respect one subgroup while breaking the other will not lift the degeneracy in the thermodynamic limit. We want to understand whether a similar mechanism is at play here for the G cluster state.

As alluded to in the main text, this is indeed the case. If we wish to protect the ground state degeneracy, one sufficient subgroup to preserve is $G_R \times \text{Rep}(G)$. This can be understood from the fact that the full algebra

$$\left\{ \overleftarrow{X}_g Z_\Gamma : g \in G, \Gamma \in \text{Rep}(G) \right\} \quad (\text{B1})$$

of a G -qudit is present at each edge, as can be seen in Eq. (43). This was also confirmed numerically in the case $G = D_3$ in Sec. VIA by forming perturbations which respect only $G_R \times \text{Rep}(G)$ and observing the stability of the phase.

However, $G_R \times \text{Rep}(G)$ may not be the only choice of preserved subgroup which protects the SPT. We can repeat the edge mode analysis for the symmetry G_L :

$$\vec{A}_g |\mathcal{C}\rangle = \left[\vec{X}_g \right] \left[\vec{X}_g \right] \left[\vec{X}_g \right] \dots \left[\vec{X}_g \right] \left[\vec{X}_g \right] \left[\vec{X}_g \right] = \left[\vec{X}_g^\dagger \right] \bullet \circ \bullet \dots \bullet \circ \bullet \left[\vec{X}_g \right],$$

and we see that we also have an algebra of X -type operators at each edge. Because Z -type operators can be combined with either left or right X -type operators to span the full algebra of a single G -qudit, this suggests that G_L may be able to protect the SPT in the case where G_R is broken. We leave further exploration of this alternative to future work. We have also not considered the role of spatial symmetries, as our perturbations were translation invariant with the periodicity of the unit cell. We leave this question to future work as well.

Appendix C: Gauging the Flux Ladder

We can proceed in much the same way as above to gauge the group-based classical Ising model, introduced in [163] as the flux ladder. The Hamiltonian of this model is given by

$$H = - \sum_i \sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_\Gamma^{(i)} \cdot Z_\Gamma^{(i+1)} \right]. \quad (\text{C1})$$

Notice that H has a global symmetry $\vec{X}_g = \prod_i \vec{X}_g^{(i)}$, but $\vec{X}_g^{(i)}$ is not a local symmetry. We can gauge this symmetry in the usual way, resulting in the Hamiltonian,

$$H_{\text{gauged}} = - \sum_i \left(\sum_{\Gamma \in \text{Rep}(G)} \text{Tr} \left[Z_\Gamma^{(i)} \cdot Z_\Gamma^{(i+1/2)} \cdot Z_\Gamma^{(i+1)} \right] + \sum_{g \in G} \vec{X}_g^{(i+1/2)} \vec{X}_g^{(i+1)} \vec{X}_g^{(i+3/2)} \right), \quad (\text{C2})$$

where the half-integer sites correspond to gauge sites inserted between the original matter sites. $\vec{X}_g^{(i)}$ is a symmetry of H_{gauged} , so we say that the gauging procedure was successful. Notice that a trivial relabeling of indices shows that H_{gauged} is equivalent to the G cluster state Hamiltonian.

Appendix D: Deriving the Fusion Category Symmetry

Consider the product of two \hat{B} operators labeled by irreps Γ_1 and Γ_2 ,

$$\hat{B}_{\Gamma_1} \hat{B}_{\Gamma_2} = \text{Tr} \left[\prod_{i \text{ even}} Z_{\Gamma_1}^{(i)} \right] \text{Tr} \left[\prod_{i \text{ even}} Z_{\Gamma_2}^{(i)} \right]. \quad (\text{D1})$$

Recall that $\text{Tr}[A]\text{Tr}[B] = \text{Tr}[A \otimes B]$, so

$$= \text{Tr} \left[\prod_{i \text{ even}} Z_{\Gamma_1}^{(i)} \otimes \prod_{i \text{ even}} Z_{\Gamma_2}^{(i)} \right]. \quad (\text{D2})$$

We also know that $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$, so

$$= \text{Tr} \left[\prod_{i \text{ even}} Z_{\Gamma_1}^{(i)} \otimes Z_{\Gamma_2}^{(i)} \right]. \quad (\text{D3})$$

The tensor product of irreps (of any linear representations, in fact) can be decomposed into a direct sum of irreps according to the fusion rules $\Gamma_i \otimes \Gamma_j = \bigoplus_k N_{\Gamma_i, \Gamma_j}^{\Gamma_k} \Gamma_k$. We then have

$$= \text{Tr} \left[\prod_{i \text{ even}} \bigoplus_k \left(Z_{\Gamma_k}^{(i)} \right)^{\oplus N_{\Gamma_1, \Gamma_2}^{\Gamma_k}} \right] = \text{Tr} \left[\bigoplus_k \prod_{i \text{ even}} \left(Z_{\Gamma_k}^{(i)} \right)^{\oplus N_{\Gamma_1, \Gamma_2}^{\Gamma_k}} \right]. \quad (\text{D4})$$

$$\text{Diagram (E2d)} \quad (E2d)$$

If we generalize the tensors in Eq. (E2a) and Eq. (E2b) from $G = \mathbb{Z}_2$ to arbitrary non-abelian G , we find

$$\text{Diagram (E3a)} \quad (E3a)$$

$$\text{Diagram (E3b)} \quad (E3b)$$

which are used to implement the duality in the main text.

Appendix F: Edge Modes from Hamiltonian

In the main text, we used the MPS form of the G cluster state to derive the edge modes. However, it is also possible to do so without referring to the explicit form of the ground state, using only the stabilizer Hamiltonian. We can rewrite the action of the global symmetries on the G cluster state using the fact that the terms of the Hamiltonian act stabilize the G cluster state, so that they can be inserted to the right of the symmetry without changing its effect on the G cluster state. Let $\hat{X}|_{|C\rangle}$ denote the effective action of an arbitrary operator \hat{X} on the G cluster state. For the X -type symmetry \overleftarrow{A}_g , we then have

$$\begin{aligned} \overleftarrow{A}_g|_{|C\rangle} &= \left(\prod_{i \text{ odd}} \overleftarrow{X}_g^{(i)} \right) \left(\overleftarrow{X}_g^{(2)} \overrightarrow{X}_g^{(3)} \overrightarrow{X}_g^{(4)} \right) \left(\overleftarrow{X}_g^{(4)} \overrightarrow{X}_g^{(5)} \overrightarrow{X}_g^{(6)} \right) \dots \\ &= \begin{cases} \overleftarrow{X}_g^{(1)} \overleftarrow{X}_g^{(2)} \mathbb{1}^{(3)} \dots \mathbb{1}^{(N-1)} \overrightarrow{X}_g^{(N)}, & N \text{ even} \\ \overleftarrow{X}_g^{(1)} \overleftarrow{X}_g^{(2)} \mathbb{1}^{(3)} \dots \mathbb{1}^{(N-2)} \overrightarrow{X}_g^{(N-1)} \overleftarrow{X}_g^{(N)}, & N \text{ odd} \end{cases} \end{aligned} \quad (F1)$$

We can apply a similar procedure to analyze the Z -type symmetry \hat{B}_Γ :

$$\begin{aligned} \hat{B}_\Gamma|_{|C\rangle} &= \text{Tr} \left[\prod_{i \text{ even}} Z_\Gamma^{(i)} \right] \Big|_{|C\rangle} \\ &= \begin{cases} \text{Tr} \left[Z_\Gamma^{(2)} \cdot Z_\Gamma^{(4)} \cdot \dots \cdot Z_\Gamma^{(N-2)} Z_\Gamma^{(N)} \right], & N \text{ even} \\ \text{Tr} \left[Z_\Gamma^{(2)} \cdot Z_\Gamma^{(4)} \cdot \dots \cdot Z_\Gamma^{(N-3)} Z_\Gamma^{(N-1)} \right], & N \text{ odd} \end{cases} \\ &= \begin{cases} \text{Tr} \left[\left(Z_\Gamma^{(1)} \cdot Z_\Gamma^{\dagger(3)} \right) \cdot \left(Z_\Gamma^{(3)} \cdot Z_\Gamma^{\dagger(5)} \right) \cdot \dots \cdot \left(Z_\Gamma^{(N-3)} \cdot Z_\Gamma^{\dagger(N-1)} \right) \cdot Z_\Gamma^{(N)} \right], & N \text{ even} \\ \text{Tr} \left[\left(Z_\Gamma^{(1)} \cdot Z_\Gamma^{\dagger(3)} \right) \cdot \left(Z_\Gamma^{(3)} \cdot Z_\Gamma^{\dagger(5)} \right) \cdot \dots \cdot \left(Z_\Gamma^{(N-4)} \cdot Z_\Gamma^{\dagger(N-2)} \right) \cdot \left(Z_\Gamma^{(N-2)} \cdot Z_\Gamma^{\dagger(N)} \right) \right], & N \text{ odd} \end{cases} \\ &= \begin{cases} \text{Tr} \left[Z_\Gamma^{(1)} \cdot Z_\Gamma^{\dagger(N-1)} \cdot Z_\Gamma^{(N)} \right], & N \text{ even} \\ \text{Tr} \left[Z_\Gamma^{(1)} \cdot Z_\Gamma^{\dagger(N)} \right], & N \text{ odd} \end{cases} \end{aligned} \quad (F2)$$

where we have used the fact that $Z_\Gamma^{(i)} \cdot Z_\Gamma^{\dagger(i)} = Z_\Gamma^{\dagger(i)} \cdot Z_\Gamma^{(i)} = \mathbb{1}_{d_\Gamma}$ because Z is unitary as well as the fact that for even i we have $Z_\Gamma^{(i)} = Z_\Gamma^{(i-1)} \cdot Z_\Gamma^{\dagger(i+1)}$ on the code space because $Z_\Gamma^{\dagger(i-1)} \cdot Z_\Gamma^{(i)} \cdot Z_\Gamma^{\dagger(i+1)}$ is a stabilizer for even i .

It is important to note that this approach has the benefit of not relying on the explicit form of the ground state, but it also has the drawback of not clearly illustrating how the edge operators reduce to an action on the edge Hilbert space, as described in the main text.

Appendix G: String Order Parameter

In this appendix, we will prove that the string order parameter introduced in Eq. (49) of the main text has zero expectation value in the $G \times \text{Rep}(G)$ -symmetric product state. We begin by proving Eq. (51) which relates the string order parameter to the delta function on the group.

Proof.

$$\begin{aligned}
\delta_{e, \bar{g}_i (\prod_{j=0}^{k-1} g_{i+1+2j}) g_{i+2k}}^G &= \sum_{\Gamma} \frac{d_{\Gamma}}{|G|} \text{Tr} \left[Z_{\Gamma}^{\dagger(i)} \cdot \left(\prod_{j=0}^{k-1} Z_{\Gamma}^{(i+1+2j)} \right) \cdot Z_{\Gamma}^{(i+2k)} \right] \\
&= \frac{|G|-1}{|G|} \hat{\mathcal{S}}^{(i, i+2k)} + \frac{d_{\mathbf{1}}}{|G|} \text{Tr} \left[Z_{\mathbf{1}}^{\dagger(i)} \left(\prod_{j=0}^{k-1} Z_{\mathbf{1}}^{(i+1+2j)} \right) \cdot Z_{\mathbf{1}}^{(i+2k)} \right] \\
&= \frac{|G|-1}{|G|} \hat{\mathcal{S}}^{(i, i+2k)} + \frac{1}{|G|} \\
\Rightarrow \hat{\mathcal{S}}^{(i, k)} &= \frac{|G|}{|G|-1} \delta_{e, \bar{g}_i (\prod_{j=0}^{k-1} g_{i+1+2j}) g_{i+2k}}^G - \frac{1}{|G|-1}
\end{aligned} \tag{G1}$$

□

We are now prepared to prove that the string order parameter has vanishing expectation value in the product state $|\psi_0\rangle$.

Proof.

$$\begin{aligned}
\hat{\mathcal{S}}^{(i, k)} |\psi_0\rangle &= \frac{|G|}{|G|-1} \delta_{g_i, g_{i+2k}} |\psi_0\rangle - \frac{1}{|G|-1} |\psi_0\rangle \\
&= \frac{1}{|G|-1} \sum_{g_i, g_{i+2k}} \delta_{g_i, g_{i+2k}} |1, e, \dots, g_i, \dots, g_{i+2k}, \dots\rangle - \frac{1}{|G|-1} |\psi_0\rangle \\
&= \frac{1}{|G|-1} \sum_{g_i} |1, e, \dots, g_i, \dots, g_i, \dots\rangle - \frac{1}{|G|-1} |\psi_0\rangle \\
\Rightarrow \langle \psi_0 | \hat{\mathcal{S}}^{(i, k)} | \psi_0 \rangle &= \frac{1}{|G|-1} \langle \psi_0 | \sum_{g_i} |1, e, \dots, g_i, \dots, g_i, \dots\rangle - \frac{1}{|G|-1} \langle \psi_0 | \psi_0 \rangle \\
&= \frac{1}{|G|-1} \langle \mathbf{1}, \mathbf{1} | \sum_{g_i} |g_i, g_i\rangle - \frac{1}{|G|-1} \\
&= \frac{1}{|G|-1} \left(\frac{|G|}{\sqrt{|G|^2}} \right) - \frac{1}{|G|-1} \\
&= 0.
\end{aligned} \tag{G2}$$

□

Appendix H: MBQC

In this appendix, we will prove two claims which we used in our discussion of universal MBQC in the main text.

1. Proof that $\vec{\Lambda}$ is Imprimitive

Consider a generic separable state in $\mathbb{C}[Q] \otimes \mathbb{C}[N]$:

$$|\psi_{QN}\rangle := \left(\sum_{q \in Q} c_q |q\rangle \right) \left(\sum_{n \in N} d_n |n\rangle \right) = \sum_{q \in Q, n \in N} c_q d_n |q, n\rangle. \tag{H1}$$

Acting on this state with $\vec{\Lambda}$, we find:

$$\vec{\Lambda} |\psi_{QN}\rangle = \sum_{q \in Q, n \in N} c_q d_n |q, \lambda(q)n\rangle. \quad (\text{H2})$$

As long as $\lambda(q)$ depends non-trivially on q – that is, there exists at least one pair $q_1, q_2 \in Q$ such that $\lambda(q_1) \neq \lambda(q_2)$ – then this state is not separable, and the gate $\vec{\Lambda}$ is imprimitive.

2. Proof of Single-Qudit Universality

In this section we will prove that the gate set R_G is universal for single G -qudit quantum computation.

Proof. Let G be a finite abelian group and consider the gate set R_G . We want to show that the lie algebra \mathfrak{a} generated by the generators of the rotations in R_G is equivalent to $\mathfrak{su}(|G|)$. We can do this by calculating commutators of elements of \mathfrak{a} and showing that this procedure produces $|G|^2$ independent Hermitian operators. We will show that there exists one independent Hermitian operator per group element besides the identity, each irrep besides the trivial irrep, and each product thereof. Along with the identity operator, this results in a total of $(|G| - 1) + (|G| - 1) + (|G| - 1)^2 + 1 = |G|^2$ independent Hermitian operators. It is clear from the form of the rotations in Eq. (76) that we already have one Hermitian operator per nontrivial group element and irrep, so we will now proceed to construct the products of these operators by evaluating commutators.

Case 1) Let $\Gamma(g) \neq 1$. This also implies that $\Gamma(\bar{g}) \neq 1$.

Case 1.1) Let $g = \bar{g}$, $\Gamma = \Gamma^\dagger$.

We have $\vec{X}_g, Z_\Gamma \in \mathfrak{a}$. We can then evaluate $[\vec{X}_g, Z_\Gamma] = (1 - \Gamma(g))\vec{X}_g Z_\Gamma$, so that $\vec{X}_g Z_\Gamma \in \mathfrak{a}$. This commutator is nonzero because we have taken $\Gamma(g) \neq 1$.

Case 1.2) Let $g = \bar{g}$, $\Gamma \neq \Gamma^\dagger$.

We have $\vec{X}_g, (Z_\Gamma + Z_\Gamma^\dagger), i(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$, so we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [\vec{X}_g, Z_\Gamma + Z_\Gamma^\dagger] &= (1 - \Gamma(g))\vec{X}_g(Z_\Gamma + Z_\Gamma^\dagger) \\ [\vec{X}_g, i(Z_\Gamma - Z_\Gamma^\dagger)] &= (1 - \Gamma(g))i\vec{X}_g(Z_\Gamma - Z_\Gamma^\dagger). \end{aligned}$$

This means that $\vec{X}_g(Z_\Gamma + Z_\Gamma^\dagger), i\vec{X}_g(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$. We see that we generate two new basis elements, which accounts for the fact that we have used both Γ and Γ^\dagger .

Case 1.3) Let $g \neq \bar{g}$, $\Gamma = \Gamma^\dagger$.

We have $(\vec{X}_g + \vec{X}_g^\dagger), i(\vec{X}_g - \vec{X}_g^\dagger), Z_\Gamma \in \mathfrak{a}$, so we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [\vec{X}_g + \vec{X}_g^\dagger, Z_\Gamma] &= (1 - \Gamma(g))(\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma \\ [i(\vec{X}_g - \vec{X}_g^\dagger), Z_\Gamma] &= (1 - \Gamma(g))i(\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma. \end{aligned}$$

Therefore, $(\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma, i(\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma \in \mathfrak{a}$. As above, we generate two new basis elements, which accounts for the fact that we have used both g and \bar{g} .

Case 1.4) Let $g \neq \bar{g}$, $\Gamma \neq \Gamma^\dagger$.

We have $(\vec{X}_g + \vec{X}_g^\dagger), i(\vec{X}_g - \vec{X}_g^\dagger), (Z_\Gamma + Z_\Gamma^\dagger), i(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$, so we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [(\vec{X}_g + \vec{X}_g^\dagger), (Z_\Gamma + Z_\Gamma^\dagger)] &= (1 - \Gamma(g))(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger) + (1 - \Gamma(\bar{g}))(\vec{X}_g^\dagger Z_\Gamma + \vec{X}_g Z_\Gamma^\dagger), \\ [i(\vec{X}_g - \vec{X}_g^\dagger), (Z_\Gamma + Z_\Gamma^\dagger)] &= i(1 - \Gamma(g))(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger) + i(1 - \Gamma(\bar{g}))(-\vec{X}_g^\dagger Z_\Gamma + \vec{X}_g Z_\Gamma^\dagger), \\ [(\vec{X}_g + \vec{X}_g^\dagger), i(Z_\Gamma - Z_\Gamma^\dagger)] &= i(1 - \Gamma(g))(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger) + i(1 - \Gamma(\bar{g}))(\vec{X}_g^\dagger Z_\Gamma - \vec{X}_g Z_\Gamma^\dagger), \\ [i(\vec{X}_g - \vec{X}_g^\dagger), i(Z_\Gamma - Z_\Gamma^\dagger)] &= -(1 - \Gamma(g))(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger) - (1 - \Gamma(\bar{g}))(-\vec{X}_g^\dagger Z_\Gamma - \vec{X}_g Z_\Gamma^\dagger). \end{aligned}$$

By summing these elements with various sets of coefficients, we can generate:

Coefficients	Sums To
$\left(\frac{1-\text{Re}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)}, 0, 0, \frac{-i\text{Im}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)} \right)$	$(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$
$\left(0, \frac{i\text{Im}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)}, \frac{1-\text{Re}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)}, 0 \right)$	$i(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger)$
$\left(0, \frac{1-\text{Re}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)}, \frac{i\text{Im}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)}, 0 \right)$	$i(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$
$\left(\frac{i\text{Im}[\Gamma(g)]}{(\Gamma(g)-1)(\Gamma(\bar{g})-1)}, 0, 0, \frac{1}{2} \left(\frac{1}{\Gamma(g)-1} + \frac{1}{\Gamma(\bar{g})-1} \right) \right)$	$(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger)$

We can therefore conclude that $(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$, $i(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger)$, $i(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$, $(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$.

Notice that when $\Gamma(g) = 1$, the commutators in **Case 1** evaluate to 0. In order to generate basis elements corresponding to these combinations, we will have to circumvent this issue by taking further commutators.

Case 2) Let $\Gamma(g) = 1$.

By the well-known fact that the intersection of the kernels of all irreducible characters of a finite group is trivial (the only $g \in G$ such that $\Gamma(g) = 1$ for all $\Gamma \in \text{irr}(G)$ is e), there exists some irrep Γ' such that $\Gamma'(g) \neq 1$ since $g \neq e$. Because in the case of abelian G the irreps form an abelian group, there also exists some Γ'' such that $\Gamma'\Gamma'' = \Gamma$. Furthermore, because $\Gamma'(g) \neq 1$ and $\Gamma'(g)\Gamma''(g) = \Gamma(g) = 1$, we know that $\Gamma''(g) \neq 1$.

Case 2.1) Let $g = \bar{g}$, $\Gamma = \Gamma^\dagger$.

Because Γ is real, Γ' and Γ'' must either both be real or both be complex. We will consider these cases separately:

Case 2.1.1) Let Γ', Γ'' real.

We know that $Z_{\Gamma''} \in \mathfrak{a}$, and we see from **Case 1** that $\vec{X}_g Z_{\Gamma'} \in \mathfrak{a}$. We can then evaluate $[\vec{X}_g Z_{\Gamma'}, Z_{\Gamma''}] = (1 - \Gamma''(g))\vec{X}_g Z_\Gamma$, so that $\vec{X}_g Z_\Gamma \in \mathfrak{a}$. This is nonzero because $\Gamma''(g) \neq 0$, and we have used the fact that $Z_{\Gamma'} Z_{\Gamma''} = Z_\Gamma$.

Case 2.1.2) Let Γ', Γ'' complex.

We know that $\vec{X}_g(Z_{\Gamma'} + Z_{\Gamma'}^\dagger)$, $i\vec{X}_g(Z_{\Gamma'} - Z_{\Gamma'}^\dagger)$, $(Z_{\Gamma''} + Z_{\Gamma''}^\dagger)$, $i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [\vec{X}_g(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= (1 - \Gamma''(g))(\vec{X}_g Z_\Gamma + \vec{X}_g Z_{\Gamma'\dagger\Gamma''}) + (1 - \Gamma''(\bar{g}))(\vec{X}_g Z_\Gamma + \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [\vec{X}_g(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))(\vec{X}_g Z_\Gamma + \vec{X}_g Z_{\Gamma'\dagger\Gamma''}) + i(1 - \Gamma''(\bar{g}))(-\vec{X}_g Z_\Gamma - \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [i\vec{X}_g(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))(\vec{X}_g Z_\Gamma - \vec{X}_g Z_{\Gamma'\dagger\Gamma''}) + i(1 - \Gamma''(\bar{g}))(-\vec{X}_g Z_\Gamma + \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [i\vec{X}_g(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= -(1 - \Gamma''(g))(\vec{X}_g Z_\Gamma - \vec{X}_g Z_{\Gamma'\dagger\Gamma''}) - (1 - \Gamma''(\bar{g}))(\vec{X}_g Z_\Gamma - \vec{X}_g Z_{\Gamma'\Gamma''\dagger}). \end{aligned}$$

By summing these four elements with appropriate coefficients $\frac{1}{4}(1, -i, -i, -1)$, we find $(1 - \Gamma''(g))\vec{X}_g Z_\Gamma$, so that $\vec{X}_g Z_\Gamma \in \mathfrak{a}$.

Case 2.2) Let $g = \bar{g}$, $\Gamma \neq \Gamma^\dagger$.

Because Γ is complex, we know that Γ' and Γ'' cannot both be real. This leads to two distinct cases:

Case 2.2.1) Without loss of generality, let Γ' be real, Γ'' be complex.

We know that $\vec{X}_g Z_{\Gamma'}$, $(Z_{\Gamma''} + Z_{\Gamma''}^\dagger)$, $i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [\vec{X}_g Z_{\Gamma'}, (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= (1 - \Gamma''(g))(\vec{X}_g Z_\Gamma + \vec{X}_g Z_\Gamma^\dagger), \\ [\vec{X}_g Z_{\Gamma'}, i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))(\vec{X}_g Z_\Gamma - \vec{X}_g Z_\Gamma^\dagger). \end{aligned}$$

This means that $\vec{X}_g(Z_\Gamma + Z_\Gamma^\dagger), i\vec{X}_g(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$.

Case 2.2.2) Let Γ', Γ'' complex.

We know that $\vec{X}_g(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i\vec{X}_g(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [\vec{X}_g(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= (1 - \Gamma''(g))\vec{X}_g(Z_\Gamma + Z_{\Gamma'\dagger\Gamma''} + Z_\Gamma^\dagger + Z_{\Gamma'\Gamma''\dagger}), \\ [\vec{X}_g(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))\vec{X}_g(Z_\Gamma + Z_{\Gamma'\dagger\Gamma''} - Z_\Gamma^\dagger - Z_{\Gamma'\Gamma''\dagger}), \\ [i\vec{X}_g(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))\vec{X}_g(Z_\Gamma - Z_{\Gamma'\dagger\Gamma''} - Z_\Gamma^\dagger + Z_{\Gamma'\Gamma''\dagger}), \\ [i\vec{X}_g(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= -(1 - \Gamma''(g))\vec{X}_g(Z_\Gamma - Z_{\Gamma'\dagger\Gamma''} + Z_\Gamma^\dagger - Z_{\Gamma'\Gamma''\dagger}). \end{aligned}$$

By summing the first and last elements with the coefficients $\frac{1}{2}(1, -1)$, we find $(1 - \Gamma''(g))\vec{X}_g(Z_\Gamma + Z_\Gamma^\dagger)$. By summing the second and third elements with the coefficients $\frac{1}{2}(1, 1)$, we find $(1 - \Gamma''(g))i\vec{X}_g(Z_\Gamma - Z_\Gamma^\dagger)$. We can therefore conclude that $\vec{X}_g(Z_\Gamma + Z_\Gamma^\dagger), i\vec{X}_g(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$.

Case 2.3) Let $g \neq \bar{g}$, $\Gamma = \Gamma^\dagger$.

Because Γ is complex, we know that Γ' and Γ'' cannot both be real. This leads to two distinct cases:

Case 2.3.1) Let Γ', Γ'' be real.

We know that $(\vec{X}_g + \vec{X}_g^\dagger)Z_{\Gamma'}, i(\vec{X}_g - \vec{X}_g^\dagger)Z_{\Gamma'}, Z_{\Gamma''} \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [(\vec{X}_g + \vec{X}_g^\dagger)Z_{\Gamma'}, Z_{\Gamma''}] &= (1 - \Gamma''(g))(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma), \\ [i(\vec{X}_g - \vec{X}_g^\dagger)Z_{\Gamma'}, Z_{\Gamma''}] &= i(1 - \Gamma''(g))(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma). \end{aligned}$$

This means that $(\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma, i(\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma \in \mathfrak{a}$.

Case 2.3.2) Let Γ', Γ'' be complex.

We know that $(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned} [(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= (1 - \Gamma''(g))((\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma + \vec{X}_g Z_{\Gamma'\dagger\Gamma''} + \vec{X}_g^\dagger Z_{\Gamma'\Gamma''\dagger}) \\ &\quad + (1 - \Gamma''(\bar{g}))((\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma + \vec{X}_g^\dagger Z_{\Gamma'\dagger\Gamma''} + \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))((\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma + \vec{X}_g Z_{\Gamma'\dagger\Gamma''} - \vec{X}_g^\dagger Z_{\Gamma'\Gamma''\dagger}) \\ &\quad + i(1 - \Gamma''(\bar{g}))((- \vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma + \vec{X}_g^\dagger Z_{\Gamma'\dagger\Gamma''} - \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))((\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma - \vec{X}_g Z_{\Gamma'\dagger\Gamma''} + \vec{X}_g^\dagger Z_{\Gamma'\Gamma''\dagger}) \\ &\quad + i(1 - \Gamma''(\bar{g}))((- \vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma'\dagger\Gamma''} + \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= -(1 - \Gamma''(g))((\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma - \vec{X}_g Z_{\Gamma'\dagger\Gamma''} - \vec{X}_g^\dagger Z_{\Gamma'\Gamma''\dagger}) \\ &\quad - (1 - \Gamma''(\bar{g}))((\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma'\dagger\Gamma''} - \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger)] &= i(1 - \Gamma''(g))((\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma + \vec{X}_g Z_{\Gamma'\dagger\Gamma''} - \vec{X}_g^\dagger Z_{\Gamma'\Gamma''\dagger}) \\ &\quad + i(1 - \Gamma''(\bar{g}))((\vec{X}_g - \vec{X}_g^\dagger)Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma'\dagger\Gamma''} + \vec{X}_g Z_{\Gamma'\Gamma''\dagger}), \\ [(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger)] &= -(1 - \Gamma''(g))((\vec{X}_g + \vec{X}_g^\dagger)Z_\Gamma + \vec{X}_g Z_{\Gamma'\dagger\Gamma''} + \vec{X}_g^\dagger Z_{\Gamma'\Gamma''\dagger}) \end{aligned}$$

$$\begin{aligned}
& - (1 - \Gamma''(\bar{g})) \left((-\vec{X}_g - \vec{X}_g^\dagger) Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right), \\
& \left[(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger) \right] = (1 - \Gamma''(g)) \left((\vec{X}_g + \vec{X}_g^\dagger) Z_\Gamma - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad + (1 - \Gamma''(\bar{g})) \left((-\vec{X}_g - \vec{X}_g^\dagger) Z_\Gamma + \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right), \\
& \left[(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \right] = i(1 - \Gamma''(g)) \left((\vec{X}_g - \vec{X}_g^\dagger) Z_\Gamma - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad + i(1 - \Gamma''(\bar{g})) \left((\vec{X}_g - \vec{X}_g^\dagger) Z_\Gamma + \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right).
\end{aligned}$$

By summing these elements with the coefficients $\frac{1}{4}(1, 0, 0, -1, 0, -1, 1, 0)$, we find $(1 - \Gamma''(g))(\vec{X}_g + \vec{X}_g^\dagger) Z_\Gamma$. By summing these elements with the coefficients $\frac{1}{4}(0, 1, 1, 0, 1, 0, 0, 1)$, we find $(1 - \Gamma''(g))i(\vec{X}_g - \vec{X}_g^\dagger) Z_\Gamma$. We can therefore conclude that $(\vec{X}_g + \vec{X}_g^\dagger) Z_\Gamma, i(\vec{X}_g - \vec{X}_g^\dagger) Z_\Gamma \in \mathfrak{a}$.

Case 2.4) Let $g \neq \bar{g}$, $\Gamma \neq \Gamma^\dagger$.

Because Γ is complex, Γ' and Γ'' must either both be real or both be complex. We will consider these cases separately:

Case 2.4.1) Without loss of generality, let Γ' be complex, Γ'' be real.

We know that $(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), Z_{\Gamma''} \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned}
& \left[(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), Z_{\Gamma''} \right] = (1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger + \vec{X}_g^\dagger Z_\Gamma + \vec{X}_g Z_\Gamma^\dagger \right), \\
& \left[i(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), Z_{\Gamma''} \right] = i(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger - \vec{X}_g^\dagger Z_\Gamma + \vec{X}_g Z_\Gamma^\dagger \right), \\
& \left[i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), Z_{\Gamma''} \right] = i(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger + \vec{X}_g^\dagger Z_\Gamma - \vec{X}_g Z_\Gamma^\dagger \right), \\
& \left[(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), Z_{\Gamma''} \right] = (1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger - \vec{X}_g^\dagger Z_\Gamma - \vec{X}_g Z_\Gamma^\dagger \right).
\end{aligned}$$

This means that $(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger), i(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger), i(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger), (\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$.

Case 2.4.2) Let Γ', Γ'' be complex.

We know that $(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \in \mathfrak{a}$. This means that we can construct the following elements of \mathfrak{a} :

$$\begin{aligned}
& \left[(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger) \right] = (1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger + \vec{X}_g Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad + (1 - \Gamma''(\bar{g})) \left(\vec{X}_g Z_\Gamma^\dagger + \vec{X}_g^\dagger Z_\Gamma + \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right), \\
& \left[(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \right] = i(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger + \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad + i(1 - \Gamma''(\bar{g})) \left(-\vec{X}_g Z_\Gamma^\dagger + \vec{X}_g^\dagger Z_\Gamma + \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right), \\
& \left[i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger) \right] = i(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad + i(1 - \Gamma''(\bar{g})) \left(-\vec{X}_g Z_\Gamma^\dagger + \vec{X}_g^\dagger Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right), \\
& \left[i(\vec{X}_g + \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \right] = -(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad - (1 - \Gamma''(\bar{g})) \left(\vec{X}_g Z_\Gamma^\dagger + \vec{X}_g^\dagger Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right), \\
& \left[i(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger) \right] = i(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger + \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
& \quad + i(1 - \Gamma''(\bar{g})) \left(\vec{X}_g Z_\Gamma^\dagger - \vec{X}_g^\dagger Z_\Gamma - \vec{X}_g^\dagger Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g Z_{\Gamma' \Gamma'' \dagger} \right),
\end{aligned}$$

$$\begin{aligned}
\left[i(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} + Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \right] &= -(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger + \vec{X}_g Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
&\quad - (1 - \Gamma''(\bar{g})) \left(-\vec{X}_g Z_\Gamma^\dagger - \vec{X}_g^\dagger Z_\Gamma - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right), \\
\left[(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), (Z_{\Gamma''} + Z_{\Gamma''}^\dagger) \right] &= (1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma + \vec{X}_g^\dagger Z_\Gamma^\dagger - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
&\quad + (1 - \Gamma''(\bar{g})) \left(-\vec{X}_g Z_\Gamma^\dagger - \vec{X}_g^\dagger Z_\Gamma + \vec{X}_g Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right), \\
\left[(\vec{X}_g - \vec{X}_g^\dagger)(Z_{\Gamma'} - Z_{\Gamma'}^\dagger), i(Z_{\Gamma''} - Z_{\Gamma''}^\dagger) \right] &= i(1 - \Gamma''(g)) \left(\vec{X}_g Z_\Gamma - \vec{X}_g^\dagger Z_\Gamma^\dagger - \vec{X}_g Z_{\Gamma' \dagger \Gamma''} + \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right) \\
&\quad + i(1 - \Gamma''(\bar{g})) \left(\vec{X}_g Z_\Gamma^\dagger - \vec{X}_g^\dagger Z_\Gamma + \vec{X}_g Z_{\Gamma' \dagger \Gamma''} - \vec{X}_g^\dagger Z_{\Gamma' \Gamma'' \dagger} \right).
\end{aligned}$$

By summing these elements with various sets of coefficients, we can generate:

Coefficients	Sums To
$\left(\frac{1 - \text{Re}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, 0, \frac{1}{4} \left(\frac{1}{\Gamma''(g) - 1} + \frac{1}{\Gamma''(\bar{g}) - 1} \right), 0, \frac{-i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, \frac{i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0 \right)$	$(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$
$\left(0, \frac{1 - \text{Re}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, \frac{1 - \text{Re}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, \frac{i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, 0, \frac{i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)} \right)$	$i(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger)$
$\left(0, \frac{i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, \frac{i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, \frac{1 - \text{Re}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, 0, \frac{1 - \text{Re}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)} \right)$	$i(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$
$\left(\frac{i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, 0, \frac{-i\text{Im}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0, \frac{1}{4} \left(\frac{1}{\Gamma''(g) - 1} + \frac{1}{\Gamma''(\bar{g}) - 1} \right), \frac{1 - \text{Re}(\Gamma''(g))}{2(\Gamma''(g) - 1)(\Gamma''(\bar{g}) - 1)}, 0 \right)$	$(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger)$

We can therefore conclude that $(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$, $i(\vec{X}_g + \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger)$, $i(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma + Z_\Gamma^\dagger)$, $(\vec{X}_g - \vec{X}_g^\dagger)(Z_\Gamma - Z_\Gamma^\dagger) \in \mathfrak{a}$.

Having demonstrated that we can construct $(|G| - 1)^2$ independent Hermitian operators, in addition to the identity and the $2|G| - 2$ operators with which we began, we can conclude that $\mathfrak{a} = \mathfrak{su}(|G|)$.